1 Introduction

QED: relativistic quantum field theory (QM + SRT) of electromagnetic interactions:

matter	$\leftrightarrow$	light
e, leptons, quark		γ

Gauge theory: gauge symmetry $U(1)$ (abelian)

- ED describes from 10^7m $\to 10^{-17}$m (SM: 10^{-18}m)
- huge variety of phenomena (physics, chemistry, ...)
- unique precision test
- prototype for theories of strong and electroweak interaction
  standard model $SU(3) \otimes SU(2) \otimes U(1)$
- important for development of QFT
  FIXME:Feynman diagram

Typical applications: high energy physics, atomic physics, positronium
Highlights:
electron magnetic moment $\mu = g\mu_B \vec{s}$, $a := \frac{g-2}{2}$
experiment: $a = (1159652180.85 \pm 7.7) \cdot 10^{-12}$ via fine structure constant
theory: $a = (1159652188.8 \pm 7.7) \cdot 10^{-12}$ via 4-loop calculation + estimate of 5-loop
Details: PRL97, 03080 $\frac{1}{2}$ (2006)
Overview

1. Introduction
2. Relativistic field theory
3. Dirac field (spin $\frac{1}{2}$ fermions)
4. Gauge principle and QED lagrangian
5. S-Matrix
6. Feynman Rules
7. Elementary QED processes (e.g. $e^+e^- \to \mu^+\mu^-$, Compton scattering)
8. Radiative corrections (g-2, renormalization)

Literature

Peskin, Schröder	QFT
Srednichi	QFT
Mandl, Shaw	QFT
Nachtmann	Elementary Particle Theory
Landau, Lifschitz	Vol4.

2 Relativistic field theory

2.1 Notation, special relativity

$\hbar = c = 1$, $\underline{\hbar c = 192.327 \text{ MeV fm}}$, $m_e = m_e c^2 = 0.511$ MeV

$e_{cgs} = \frac{e}{\sqrt{4\pi}}$, $A_{cgs} = \sqrt{4\pi} A$,
div $\vec{E}_{cgs} = 4\pi\rho_{cgs} \quad \rightarrow \quad \sqrt{4\pi}$ div $\vec{E} = 4\pi \frac{1}{\sqrt{4\pi}} \rho$

$$\begin{array}{ll} \nabla \cdot \vec{E} = \rho & \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} - \partial_t \vec{E} = \vec{j} & \nabla \times \vec{E} + \partial_t \vec{B} = 0 \end{array}$$

$\alpha_{cgs} \equiv \frac{1}{137} = \frac{e_{cgs}^2}{\hbar c} \rightarrow \alpha = \frac{e^2}{4\pi}$

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ \\ \vec{x} \\ \end{pmatrix}; \qquad g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$g^\mu_\nu = \delta^\mu_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad x_\mu = g_{\mu\nu} x^\nu = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix}$$

$x^2 = x \cdot x = x^\mu x_\mu = x^\mu x^\nu g_{\mu\nu} = \left(x^0\right)^2 - \vec{x}^2$
invariant under Lorentz transformations
$x^2 = 0 \Leftrightarrow \vec{x}^2 = t$ describes light

Lorentz Trafo:

contravariant: $x'^\mu = \Lambda^\mu_\rho x^\rho$
covariant: $x'_\mu = \Lambda^\nu_\mu x_\nu$
invariance: $x'^2 = x'^\mu x'^\nu g_{\mu\nu} = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho x^\sigma \stackrel{!}{=} x^\rho x^\sigma g_{\rho\sigma}$

$$\Rightarrow \boxed{\Lambda^\mu_\rho \Lambda^\nu_\sigma g_{\mu\nu} = g_{\rho\sigma}}$$

$\Lambda^\mu{}_\rho \equiv \Lambda$, $\Lambda^\mu_\rho = \Lambda^{T\mu}_\rho = \left(g \Lambda^T g\right)_\rho{}^\mu$, $g_{\mu\nu} \equiv g$

$$\Rightarrow \left(g \Lambda^T g\right)^\rho{}_\nu \Lambda^\nu{}_\sigma = g^\rho{}_\sigma \Leftrightarrow g \Lambda^T g \Lambda = 1$$

$\Rightarrow \boxed{\Lambda^T g \Lambda = g}$ metric invariant under Lorentz trafo
$\Rightarrow$ det $\Lambda = \pm 1$

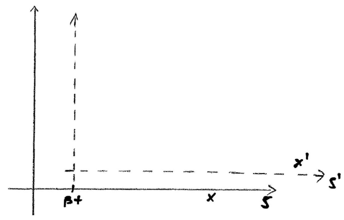
homogene Lorentz group: $X'^\mu = \Lambda^\mu{}_\nu x^\nu$
Poincaré: group: $X'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

Examples:

$$\Lambda = \left(\begin{array}{c|c} 1 & \\ \hline & R_{3\times 3} \end{array}\right) \qquad R^T R = \mathbb{1}_{3\times 3} \qquad \text{spatial rotation}$$

$$\Lambda_p = \left(\begin{array}{c|c} 1 & \\ \hline & -\mathbb{1} \end{array}\right) \qquad \text{spatial reflection}$$

$$\Lambda_T = \left(\begin{array}{c|c} -1 & \\ \hline & \mathbb{1} \end{array}\right) \qquad \text{time reflection}$$



$$\Lambda^\mu{}_\nu \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \;\widehat{=}\; \begin{array}{l} t' = \gamma(t - \beta x) \\ x' = \gamma(x - \beta t) \end{array} \qquad \text{Lorentz boost in } \hat{x}\text{-direction}$$

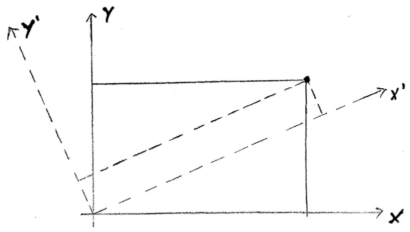
4-Momentum: $p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$, $p'^\mu = \Lambda^\mu{}_\nu p^\nu$

Invariants: $p^\mu p_\mu = p^2 = m^2,\ (p_1 + p_2)^2,\ p_1 \cdot p_2,\ p \cdot x,\ e^{ip\cdot x}$

Fields and derivatives

Scalar field $\phi'(x') = \phi(\Lambda^{-1}x') = \phi(x)$

Vector field $A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$



$$\frac{\partial \phi'(x')}{\partial x'^\mu} = \frac{\partial \phi(x)}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = \Lambda_\mu{}^\nu \frac{\partial \phi(x)}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu \phi$$

$$\Lambda_{\nu\rho}\Lambda^\nu{}_\sigma = g_{\rho\sigma}; \quad x'^\alpha = \Lambda^\alpha{}_\beta x^\beta \to \Lambda_{\alpha\nu} x'^\alpha = \underbrace{\Lambda_{\alpha\nu}\Lambda^\alpha{}_\beta}_{g_{\nu\beta}} x^\beta = x_\nu$$

$$\Rightarrow x^\nu = \Lambda_\alpha{}^\nu x'^\alpha; \quad \frac{\partial x^\nu}{\partial x'^\mu} = \Lambda_\alpha{}^\nu \delta^\alpha{}_\mu = \Lambda^\nu_\mu$$

In general:

$$\text{contravariant} \left\{ \begin{array}{c} \text{tensor} \\ \text{pseudotensor} \end{array} \right\} \text{of rank n}$$

$$T'^{\mu_1 \ldots \mu_n}(x') = \left\{ \begin{array}{c} 1 \\ \det \Lambda \end{array} \right\} \Lambda^{\mu_1}{}_{\nu_1} \ldots \Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1 \ldots \nu_n}(x)$$

invariant tensor $g^{\mu\nu} = g_{\mu\nu}$ $\epsilon^{\mu\nu\rho\sigma}$ totally antisymmetric

$\epsilon^{0123} = -1$

invariant operator $\partial^\mu \partial_\mu \phi - \left(\partial_0^2 - \vec{\partial}^2\right)\phi = \Box\phi$ d'Alembert operator
z.B. $\Box\phi = 0$ für freie Ausbreitung

2.2 Lagrangian formulation, quantization

Analogy:
mechanical system $q_i(t)$; [discrete] $\delta_{ij}, \sum_{i=1}^N$

field theory $\phi(t,\vec{x})$ $\quad \vec{x}\left[\begin{array}{c}\text{continously}\\ \text{infinite}\end{array}\right]\delta(\vec{x}-\vec{y}), \int \mathrm{d}^3x$

Example: real scalar field $\phi(t,\vec{x}) \equiv \phi(x)$[e.g. mesons, higgs boson, ...]
Assumption: $\mathcal{L} = \mathcal{L}(\phi, \delta\phi)$ Lagrangian density
Action: $S(\Omega) = \int_\Omega \mathrm{d}^4x\mathcal{L}(\phi,\partial\phi) = \int \mathrm{d}t \underbrace{\int \mathrm{d}^3x\mathcal{L}(\phi,\delta\phi)}_{L}$

Action principle: variation $\phi(x) \to \tilde{\phi}(x) = \phi(x) + \delta\phi(x)$, $\delta\phi(x) = 0$, $x \in \Gamma(\Omega)$ (boundary) $\Rightarrow \delta\partial_\mu\phi \equiv \partial_\mu\tilde{\phi} - \partial_\mu\phi = \partial\mu(\tilde{\phi}-\phi) = \partial_\mu\delta\phi$

$$\begin{aligned}\delta S(\Omega) &= \int_\Omega \mathrm{d}^4x\left(\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta L}{\delta\partial_\mu\phi}\delta\partial_\mu\phi\right)\\ &= \int \mathrm{d}^4x\left(\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta L}{\delta\partial_\mu\phi}\right)\delta\phi + \underbrace{\int_\Omega \mathrm{d}^4x\partial_\mu\left(\frac{\partial\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi\right)}_{=0\ \delta\,\phi|_\Gamma=0}\\ &\stackrel{!}{=} 0\end{aligned}$$

$$\forall\delta\phi \Rightarrow \boxed{\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} = 0}$$

field canoncially conjugate to $\phi(x) : \pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\phi}$
Hamiltonian density $\mathcal{H} = \pi(x)\partial_0\phi(x) - \mathcal{L}$ no Lorentz scalar

$$H = \int \mathrm{d}^3x\mathcal{H}$$

example: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \Rightarrow \left(\delta^\mu\delta_\mu + m^2\right)\phi = 0$ Klein-Gordon-equation
$\pi(x) = \partial_0\phi(x) \Rightarrow \mathcal{H} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}\left(\vec{\partial}\phi\right)^2 + \frac{1}{2}m^2\phi^2$ canonical field quantization

$[\pi(t,\vec{x}),\phi(t,\vec{y})] = -i\delta(\vec{x}-\vec{y})$ Heisenberg commutator
$[\phi(t,\vec{x}),\phi(t,\vec{y})] = 0 = [\pi(t,\vec{x}),\pi(t,\vec{y})]$ relations (equal time)

Generalization: several real scalar fields $\phi(x)$
$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i\phi_i \Rightarrow (\partial^2 + m^2)\phi_i = 0\forall i = 1,\dots,m$
$[\pi_i(t,\vec{x}),\phi_j(t,\vec{y})] = -i\delta_{ij}\delta(\vec{x}-\vec{y})$
complex KleinGordon field: $\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi$

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}},\ \phi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}},\ \phi^\dagger\phi \text{ independant fields}$$

2.3 Symmetries and conservation laws (Noether's theorem)

1. Translation invariance ($\mathcal{L} = \mathcal{L}\left(\phi(x),\partial\phi(x)\right)$; i.e. no explicit x-dependance)

$$\begin{aligned}\Rightarrow \partial_\alpha g^{\alpha\beta}\mathcal{L} \equiv \partial^\beta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\partial^\beta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\alpha\phi}\partial^\beta\partial_\alpha\phi\\ &= \partial_\alpha\frac{\delta\mathcal{L}}{\delta\partial_\alpha\phi}\partial^\beta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\alpha\phi}\partial_\alpha\partial^\beta\phi\\ &\Rightarrow \partial_\alpha\left[\frac{\delta\mathcal{L}}{\delta\partial_\alpha\phi}\partial^\beta\phi - g^{\alpha\beta}\mathcal{L}\right] = 0\end{aligned}$$

$$\Rightarrow \partial_\alpha\mathcal{T}^{\alpha\beta} = 0 \quad \mathcal{T}^{\alpha\beta} = \frac{\delta\mathcal{L}}{\delta\partial_\alpha\phi}\partial^\beta\phi - g^{\alpha\beta}\mathcal{L} \text{ energy-momentum tensor}$$

general continuity equation: $\partial_\alpha S^\alpha = \partial_t S^0 + \vec{\partial}\vec{S} = 0$
$\rightarrow$ conserved quantity $\int \mathrm{d}^3xS^0$
$\int \mathrm{d}^3x\mathcal{T}^{0\beta} = P^\beta,\ \mathcal{T}^{00} = \pi(x)\partial_0\phi(x) - \mathcal{L} = \mathcal{H}$

2. internal symmetries, $U(1)$ invariance, charge conservation

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi\phi \to \phi' \qquad = e^{i\epsilon}\phi = (1+i\epsilon)\phi\phi^\dagger \to \phi'^\dagger = e^{-i\epsilon}\phi^\dagger = (1-i\epsilon)\phi$$

In general:

$$\begin{aligned}\phi\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi_i}\delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\delta\partial_\mu\phi_i \\ &= \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\partial_\mu\delta\phi_i \\ &= \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\delta\phi_i\right] \\ &= {}^*0\end{aligned}$$

⇒ conserved current
Complex Klein-Gordon-Field: $\{\phi_i\} = (\phi, \phi^\dagger)$

$$\begin{aligned}\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\delta\phi_i\right] &= \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^\dagger}\delta\phi^\dagger \\ &= i\epsilon\underbrace{\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^\dagger}\phi^\dagger\right)}_{=:S^\mu}\end{aligned}$$

$\partial_\mu S^\mu = 0$; $S^\mu = \partial^\mu\phi^\dagger\phi - \partial^\mu\phi\phi^\dagger \to$ conserved quantity $\int \mathrm{d}^3 x S^0$ charge; e.g. electrical charge

2.4 Solution of the Klein-Gordon equation, particle interpretation

1 real field ϕ

KG:$\left(\delta^2 + m^2\right)\phi = 0$ Ansatz: $\phi(x) \sim e^{\pm ik\cdot x}$ with $k = (\omega, \vec{k})$, $k\cdot x = \omega t - \vec{k}\cdot\vec{x}$ plane wave ansatz. $\to -k^2 + m^2 = -\omega^2 + \vec{k}^2 + m^2 = 0$ General solution: ϕ real; $\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\Big[\underbrace{a(k)}_{\text{i.e.}a(\vec{k})\ \omega=f(\vec{k})} \quad e^{-ik\cdot x} + \underbrace{a^\dagger(k)}_{\dagger \text{ guarantees } \phi \text{ to be real}} \quad e^{ik\cdot x}\Big]\frac{1}{2m}$

$k^\circ = \omega$ $\phi^+ = \phi$
canonical commutation rules: $[\pi(t,\vec{x}), \phi(t,\vec{y})] = -i\delta^{(3)}(\vec{x}-\vec{y}), ...$
$\Leftrightarrow$ $[a(k), a^+(k')] = (2\pi)^3 2\omega_k \delta^3\left(\vec{k} - \vec{k}'\right)$; $[a(k), a^+(k')] = [a^+(k), a^+(k')] = 0$ analogous to harmonic oscillator

Fock space

vacuum:	$\lvert 0\rangle$ with $a(k)\lvert 0\rangle = 0$
1-particle state:	$\lvert k\rangle = a^+(k)\lvert 0\rangle$
2-particle state:	$\lvert k_1, k_2\rangle = a^+(k_1)a^+(k_2)\lvert 0\rangle = a^+(k_2)a^+(k_1)\lvert 0\rangle = \lvert k_2, k_2\rangle$ 2 free particles, state symmetric → indistinguishable particles, Bose particles
normalization:	$\langle 0 \mid 0\rangle = 1$ $\quad \langle k \mid k'\rangle = \langle 0\rvert a(k)a^+(k')\lvert 0\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k}-\vec{k}')$

Energy

H= $\int$ d$^3x\mathcal{H}$, $\mathcal{H} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}\left(\vec{\partial}\phi\right)^2 + \frac{1}{2}m^2\phi^2$

reformulating $\mathcal{H}$ via a, a^+ leads to: H=$\displaystyle\int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\omega_k \frac{1}{2}\left[a^+(k)a(k) + a(k)a^+(k)\right]$ $\langle 0| H |0\rangle = \displaystyle\int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\omega_k \frac{1}{2}\Big[(2\pi)^3 2\omega_$

$$\delta(\vec{k}-\vec{k}') = \int \frac{\mathrm{d}^3x}{(2\pi)^3} e^{-i(\vec{k}-\vec{k}')\vec{x}} \xrightarrow{\vec{k}=\vec{k}'} \delta(0) = \int \frac{\mathrm{d}^3x}{(2\pi)^3} = \frac{V}{(2\pi)^3}$$

finite Volume $V = L^3(\to\infty)$ counting the number of states periodic boundary conditions: $e^{ik^1L} = e^{2i\pi n}k^1 = \frac{2n\pi}{L}$ 1 sate per $\left(\frac{2\pi}{L}\right)^3$ ($\vec{k}$-space) # of states : $\frac{V}{(2\pi)^3}$ d$^3k \Rightarrow \langle 0| H |0\rangle = \underbrace{\displaystyle\int \frac{V\,\mathrm{d}^3k}{(2\pi)^3} \quad \underbrace{\frac{1}{2}\omega_k}_{\widehat{=}\omega\cdot\frac{\hbar}{2}\text{ ground state energy of HO}}}_{\text{energy oscillator has } \frac{\omega_k}{2} \text{ summation over inf. many}}$ zer

<u>subtraction</u> H=$\int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\omega_k a^+(k)a(k)$ normal ordering: $: a^+(k)a(k) + a(k)a^+(k) := 2a^+(k)a(k)$ $: a(k_1)a^+(k_2)a(k_3) := a^+(k_2)a(k_1)a(k_3)$

$$\begin{aligned} H = |k\rangle &= \int \frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega'_k}\omega'_k a^+(k') \underbrace{a(k')a^+(k)}_{(2\pi)^3 2\omega_k\delta(\vec{k}-\vec{k})} |0\rangle \\ &= \omega_k a^+(k)|0\rangle = \omega_k |k\rangle \end{aligned}$$

analogously: $\vec{p} = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\vec{k}a^+(k)a(k)$ $N = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}a^+(k)a(k)$; $\int_k := \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}$

complex scalar field

$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}[a(k)e^{-ik\cdot x} + b^+(k)e^{ik\cdot x}]$ $[a(k), a^+(k')] = [b(k), b^+(k')] = (2\pi)^3 2\omega_k\delta(\vec{k}-\vec{k}')$, zero otherwise $j^\mu = -i(\delta^\mu\phi^+\phi - \delta^\mu\phi\phi^+)$ $Q = \int$ d$^3xj^\circ = \int_k a^+(k)a(k) - b^+(k)b(k)$ (after normal ordering) → particle/antiparticle: a-type/ b-type, same mass but charge differs by a sign

2.5 Covariant commutators, Feynman propagator

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\left[a(k)e^{-ik\cdot x} + a^+(k)e^{ik\cdot x)}\right] \equiv \phi_+(x) + \phi_0(x)$$

$-ik\cdot x = -i\omega t + i\vec{k}\cdot\vec{x}$ $E \mathrel{\widehat{=}} i\delta_t \to \omega$ $[\phi(x), \phi(y)]\,|_{x^0=y^0} = 0$ equal time commutator

$$\begin{aligned} [\phi(x), \phi(y)] &= \underbrace{[\phi_+(x), \phi_-(y)]}_{=:i\Delta_+(x-y)} + \underbrace{[\phi_-(x), \phi_+(y)]}_{=:i\Delta_-(x-y)} \\ &= i\Delta_+(x-y) - i\Delta_+(y-x) =: i\Delta(x-y) \\ i\Delta_+(x-y) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\int \frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega_k} e^{-ik\cdot x}e^{ik'\cdot y} \underbrace{(2\pi)^3 2\omega\delta(\vec{k}-\vec{k}')}_{[a(k),a^+(k')]} \\ &= \int_k e^{-ik\cdot(x-y)} \end{aligned}$$

$\Rightarrow i\Delta(x-y) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega}\left(e^{-ik(x-y)} - e^{ik(x-y)}\right) = -i\int \frac{\mathrm{d}^3k}{(2\pi)^3\omega}\sin(k\cdot(x-y))$ Properties:

1. $(\Box_x + m^2)\Delta(x-y) = 0$
2. $[\phi(x), \phi(y)] = i\Delta(x-y)$ only function of x -y → translation invariance

†first example where "renormalization" is needed; Reason: infinitely many degrees of freedom

3. $[\phi(x),\phi(y)]|_{x^\circ=y^\circ} = i\Delta(0,\vec{x}-\vec{y}) = i\int \underbrace{\frac{\mathrm{d}^3k}{(2\pi)^3\omega}}_{\text{even in }\vec{k}} \underbrace{\sin\left(\vec{k}\cdot(\vec{x}-\vec{y})\right)}_{\text{odd in }\vec{k}} = 0$

4.

$$\begin{aligned}[\pi(x),\phi(y)] &= \frac{\delta}{\delta x_0}[\phi(x),\phi(y)] = i\frac{\delta}{\delta x_0}\Delta(x-y)\\ &= \frac{-i}{(2\pi)^3}\int \mathrm{d}^3k\cos\left(\omega(x_0-y_0)\right)e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \overset{x_0=y_0}{=} -i\delta(\vec{x}-\vec{y})\end{aligned}$$

5. $\frac{\mathrm{d}^3k}{2\omega}$ is Lorentz invariant $\Rightarrow \Delta(x'-y') = \Delta(x-y)$ invariant

6. $\{c)\wedge e)\} \Rightarrow [\phi(x),\phi(y)] = 0$ for $(x-y)^2 < 0$ causality

$$\Delta_\pm(x) = \frac{-1}{(2\pi)^4}\int_{C^\pm\times\mathbb{R}^3} \mathrm{d}^4k\frac{e^{-ik\cdot x}}{\underbrace{k^2-m^2}_{k_0^2-\omega^2}} \qquad \omega = \sqrt{\vec{k}^2+m^2}$$

$\int_{C^+}\frac{\mathrm{d}k_0e^{-ik_0t}}{(k_0-\omega)(k_0+\omega)} = 2\pi i\,\mathrm{Res}\left[\frac{e^{-ik_0t}}{(k_0-\omega)(k_0+\omega)}\right]|_{k_0=\omega} = \frac{2\pi i}{2\omega}e^{-i\omega t} \Rightarrow \Delta_+(x) = -i\int\frac{\mathrm{d}^3k}{(2\pi)^32\omega}e^{-ikx};\ k_0 = \omega$

$$\begin{aligned}i\Delta_+(x-x') &= \langle 0|\,[\phi_+(x),\phi_-(x')]\,|0\rangle\\ &= \langle 0|\,\phi_+(x),\phi_-(x')\,|0\rangle\\ &= \langle 0|\,\phi(x),\phi(x')\,|0\rangle\end{aligned}$$

Feynman propagator

T product (time ordered product):

$$T\phi(x)\phi(x') = \Theta(t-t')\phi(x)\phi(x') + \Theta(t'-t)\phi(x')\phi(x) = \begin{cases}\phi(x)\phi(x'), & t>t'\\ \phi(x')\phi(x), & t'>t\end{cases}$$

Definition: $\langle 0|\,T\phi(x)\phi(x')\,|0\rangle = i\Delta_F(x-x')$

$$\begin{aligned}i\Delta_F(x) &= \Theta(t)\langle 0|\,\phi(x)\phi(0)\,|0\rangle + \Theta(-t)\langle 0|\,\phi(0)\phi(x)\,|0\rangle\\ &= \Theta(t)i\Delta_+(x) + \Theta(-t)i\underbrace{\Delta_+(-x)}_{-\Delta_-(x)}\end{aligned}$$

$\Rightarrow \Delta_F(x) = \Theta(t)\Delta_+(x) - \Theta(-t)\Delta_-(x) = \pm\Delta_\pm(x), t\in\pm\mathbb{R}$
Integral Representation: $\Delta_F(x) = \frac{1}{(2\pi)^4}\int_{C_F}\mathrm{d}^4k\frac{e^{-ikx}}{k^2-m^2} \qquad \omega\to(\omega-i\eta)$
$k^2-m^2 = k_0^2-\vec{k}^2-m^2 = k_0^2-\omega^2+2i\omega\eta = k_0^2-\omega^2+i\epsilon = k^2-m^2+i\epsilon$
$\Rightarrow \Delta_F(x) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2-m^2+i\epsilon}$
$\Rightarrow (\delta^2+m^2)\Delta_F(x)^\ddagger = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-k^2+m^2}{k^2-m^2+i\epsilon}e^{-ikx} = -\delta^{(4)}(x)$ $(\delta^2+m^2)\phi(x) = j(x) \to \phi(x) = -\int\mathrm{d}^4y\Delta_F(x-y)j(y)$ $i\Delta_F(x) = \langle 0|\,T\phi(x)\phi(0)\,|0\rangle = \int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ikx}\frac{i}{k^2-m^2+i\epsilon}$

1. Green's Function of the Klein-Gordon equation δ^2+m^2
2. correlation function (time ordered)
3. Feynman propagator $\underbrace{\langle 0|}_{\text{vacuum}}\ T\ \underbrace{\phi x}_{\text{annihilation at x}}\ \underbrace{\phi 0}_{\text{creation at 0}}\ \underbrace{|0\rangle}_{\text{vacuum}}$

‡Green's function of the Klein-Gordon operator