28a) $\partial_z^k(e^{-z}) = (-)^k e^{-z}$,

$$\partial_z^k(z^l) = \begin{cases} l\,\partial_z^{k-1}(z^{l-1}) = \ldots = \dfrac{l!}{(l-k)!} z^{l-k}, & k \le l, \\ 0 & k > l. \end{cases}$$

$$\partial_z^k(AB) = \sum_{m=0}^{k} \binom{k}{m} (\partial_z^m A)(\partial_z^{k-m} B).$$

28b) First we compute: $L_p = e^z \partial_z^p (z^p e^{-z}) =$

$$= e^z \sum_{l=0}^{p} \binom{p}{l} \partial_z^l (z^p)\, \partial_z^{p-l}(e^{-z}) =$$

$$= e^z \sum_{l=0}^{p} \frac{p!}{(p-l)!\,l!} \frac{p!\,z^{p-l}}{(p-l)!} (-)^{p-l} e^{-z} = \sum_{l=0}^{p} \frac{(-)^{p-l} p!^2}{(p-l)!^2\, l!} z^{p-l}$$

With this:

$$L_p^k = \partial_z^k L_p = \sum_{l=0}^{p-k} \frac{(-)^{p-l} p!^2}{(p-l)!^2\, l!} \frac{(p-l)!}{(p-l-k)!} z^{p-l-k}$$

Set $m = p-l-k \Leftrightarrow l = p-k-m$, then:

$$L_p^k = \sum_{m=0}^{p-k} \frac{(-)^{k+m} p!^2}{(p-k-m)!\,(k+m)!\,m!} z^m .$$

28c) We compute the first 2 two terms of the diff. equ. in term:

$$z\,\partial_z^2 L_p^k = \sum_{m=0}^{p-k} \frac{(-)^{k+m} p!^2\, m(m-1)}{(p-k-m)!\,(k+m)!\,m!} z^{m-1}$$

Because of the factor m, we can start the sum at $m=1$, instead of 0, and divide m out of $m!$. Then set $n = m-1$, gives:

$$z\,\partial_z^2 L_p^k = \sum_{n=0}^{p-k-1} \frac{(-)^{k+n+1} p!^2\, n}{(p-k-n-1)!\,(k+n+1)!\,n!} z^n \qquad n \to m$$

$$= \sum_{m=0}^{p-k-1} \frac{(-)^{k+m} p!^2\, z^m}{(p-k-m)!\,(k+m+1)!\,m!} \left[-m(p-k-m)\right]$$

The last rewriting is such that the structure of L_p^k can be reconstructed easily without having to deal with ratios. Similarly:

$$(k+1-z)\,\partial_z L_p^k = (k+1-z) \sum_{m=0}^{p-k} \frac{(-)^{k+m} p!^2\, m\, z^{m-1}}{(p-k-m)!\,(k+m)!\,m!} =$$

$$= \sum_{m=1}^{p-k} \frac{(-)^{k+m} p!^2 (k+1) z^{m-1}}{(p-k-m)!\,(k+m)!\,(m-1)!} - \sum_{m=0}^{p-k-1} \frac{(-)^{k+m} p!^2\, m\, z^m}{(p-k-m)!\,(k+m)!\,m!} - \frac{(-)^p p!^2 (p-k) z^{p-k}}{p!\,(p-k)!}$$

In the first term $n = m-1$ (and after that replace $n \to m$):

$$= \sum_{m=0}^{p-k-1} \frac{(-)^{k+m} p!^2\, z^m}{(p-k-m)!\,(k+m+1)!\,m!} \left[-(k+1)(p-k-m) - m(k+m+1)\right] - \frac{(-)^p p!^2 (p-k) z^{p-k}}{p!\,(p-k)!}$$

Combining the terms in the square brackets [...] gives

$$-m(p-k-m) - (k+1)(p-k-m) - m(k+m+1) =$$

$$= -(p-k-m+m)(k+m+1) = -(p+k)(k+m+1)$$

Summing $z\partial_z^2 L_p^k$ and $(k+1-z)\partial_z$ using this, gives:

28 cont) $\left[z\partial_z^2 + (k+1-z)\partial_z\right] L_p^k =$

$$= \sum_{m=0}^{p-k-1} \frac{(-)^{k+m}\, p!^2\, z^m\,(k+m+1)}{(p-k-m)!\,(k+m+1)!\,m!}\left[-(p+k)\right] + \left[-(p+k)\right]\frac{(-)^p\, p!^2\, z^{p-k}}{p!\,(p-k)!}$$

$$= -(p+k)\sum_{m=0}^{p-k}\frac{(-)^{k+m}\,p!^2}{(p-k-m)!\,(k+m)!\,m!} = -(p+k)L_p^k.$$

29a) Substitute $x = \beta z$, and denote $\psi(x) = \psi(z)$:

$$1 = \left[\frac{\hbar^2}{2m\beta^2}\partial_z^2 + \frac{e^2}{\beta|z|} + E\right]\psi(z) \doteq \frac{\hbar^2}{2m\beta^2}\left[\partial_z^2 - \frac{1}{4} + \frac{\alpha}{|z|}\right]$$

$$\Rightarrow E = -\frac{1}{4}\frac{\hbar^2}{2m\beta^2}, \quad \frac{e^2}{\beta} = \frac{\hbar^2}{2m\beta^2}\alpha \Rightarrow \beta = \frac{\hbar^2}{2me^2}\alpha$$

$$E = -\frac{1}{4}\frac{\hbar^2}{2m}\left(\frac{2me^2}{\hbar^2}\frac{1}{\alpha}\right)^2 = -\frac{me^4}{2\hbar^2}\frac{1}{\alpha^2}$$

29b) For $z \to \infty$ the $\frac{\alpha}{|z|}$-term can be neglected. The solutions of $\left[\partial_z^2 - \frac{1}{4}\right]\psi(z) = 0$ are $\psi_\pm(z) = e^{\pm z/2}$. Since we are looking for bounded states, we can only take $\psi = \psi_+ = e^{-z/2}$.

29c) $\psi(z) = g(z)\varphi(z)$, $g(z) = z e^{-z/2}$

$$\frac{g'}{g} = \frac{-\frac{1}{2}z e^{-z/2} + e^{-z/2}}{z e^{-z/2}} = \left(\frac{1}{z} - \frac{1}{2}\right), \quad \left(\frac{g'}{g}\right)' = -\frac{1}{z^2}$$

Also $\left(\frac{g'}{g}\right)' = \frac{g''}{g} - \left(\frac{g'}{g}\right)^2 \Rightarrow \frac{g''}{g} = \left(\frac{g'}{g}\right)' + \left(\frac{g'}{g}\right)^2$

Hence: $\frac{g''}{g} = \left(\frac{1}{z} - \frac{1}{2}\right)^2 - \frac{1}{z^2} = -\frac{1}{z} + \frac{1}{4}.$

Compute:

$$\frac{1}{g}\partial_z^2\psi = \frac{1}{g}\partial_z^2(g\varphi) = \left[\frac{g''}{g} + 2\frac{g'}{g}\partial_z + \partial_z^2\right]\varphi$$

$$= \left[\partial_z^2 + 2\left(\frac{1}{z} - \frac{1}{2}\right)\partial_z + \frac{1}{4} - \frac{1}{z}\right]\varphi.$$ Inserting gives:

$$z\left[\partial_z^2 + \left(\frac{2}{z} - 1\right)\partial_z + \frac{1}{4} - \frac{1}{z} - \frac{1}{4} + \frac{\alpha}{z}\right]\varphi = 0$$

$$\left[z\partial_z^2 + (2 - z)\partial_z - (1-\alpha)\right]\varphi = 0$$

We read off: $2 = k+1$, $p - k = -1 + \alpha$

$\Rightarrow k = 1$, $p = \alpha$

29d) Because the differential operator $\left[-\frac{\hbar^2}{2m}\partial_x^2 - \frac{e^2}{|x|}\right]$ is symmetric under $x \to -x$, it follows that if $\psi(x)$ is a solution, so is $\psi(-x)$.

The bounded solutions are:

$$\psi(x) = \begin{cases} zL_p^1(z)\,e^{-z/2}A_+ & z > 0 \\ (-z)L_p^1(-z)\,e^{+z/2}A_- & z < 0 \end{cases}$$

At $x = 0$ the solution should be continuously differentiable: $\psi(-\varepsilon) = \psi(+\varepsilon) = 0$, $\varepsilon \to 0$.

29d cont) Also because of the prefactor z:

$$\left(\begin{array}{l}\psi'(z) = L_p'(0)A_+ \\ \psi'(-z) = L_p'(0)A_-\end{array}\right. \Rightarrow A_+ = A_- = A.$$

29e) So the eigenstates are:

$$\psi(z) = A_p |z| L_p'(|z|) e^{-|z|/2}.$$

Note that $p \geq 1$ because $L_0' = d_z(-1) = 0$,

hence the spectrum is: $E_p = -\frac{me^4}{2\hbar^2}\frac{1}{p^2}$.

30a) $\vec{r}, \vec{p}$ are Hermitean. $\vec{L}$ is also Hermitean:

$$L_i^\dagger = (\epsilon_{ijk} x_j p_k)^\dagger = \epsilon_{ijk} p_k x_j = \epsilon_{ijk}(x_j p_k - [x_j, p_k])$$
$$= L_i - \epsilon_{ijk} i\hbar \delta_{jk} = L_i \checkmark$$

For $\vec{A}, \vec{B}$ Hermitean: $(\vec{A}\times\vec{B})^\dagger = -\vec{B}\times\vec{A}$

$$(\epsilon_{ijk} A_j B_k)^\dagger = \epsilon_{ijk} B_k A_j = \epsilon_{ikj} B_j A_k = -\epsilon_{ijk} B_j A_k$$

Hence:

$$\vec{R} = \frac{1}{2m}(\vec{L}\times\vec{p} - \vec{p}\times\vec{L}) + k\frac{\vec{r}}{r} \text{ Hermitean.}$$

30b) Let $\vec{A}, \vec{B}$ be two operators that:

$$[L_i, A_j] = i\hbar\epsilon_{ijk} A_k, \quad [L_i, B_j] = i\hbar\epsilon_{ijk} B_k$$

then:

$$[L_i, \epsilon_{jmn} A_m B_n] = \epsilon_{jmn}([L_i, A_m] B_n + A_m [L_i, B_n])$$
$$= i\hbar\epsilon_{jmn}(\epsilon_{imp} A_p B_n + \epsilon_{inp} A_m B_p)$$
$$= i\hbar(\epsilon_{imp}\epsilon_{jnp} + \epsilon_{inp}\epsilon_{jmp}) A_m B_n$$
$$= i\hbar(\cancel{\delta_{ij}\delta_{mn}} - \delta_{in}\delta_{jm} - \cancel{\delta_{ij}\delta_{mn}} + \delta_{im}\delta_{jn}) A_m B_n$$
$$= i\hbar\epsilon_{ijk}\epsilon_{kmn} A_m B_n \Rightarrow [L_i, (\vec{A}\times\vec{B})_j] = i\hbar\epsilon_{ijk}(\vec{A}\times\vec{B})_k.$$

Since $\vec{R}$ is built out of $\vec{L}\times\vec{p}$, $\vec{p}\times\vec{L}$, and $\vec{r}/r$, they all transform as a vector, ($\frac{1}{r} = \frac{1}{\sqrt{r^2}}$, and r^2 is invariant by 25d) hence result follows

30c) $$[H, \vec{R}] = \left[\frac{1}{2m}\vec{p}^2 - \frac{k}{r}, \frac{1}{2m}(\vec{L}\times\vec{p} - \vec{p}\times\vec{L}) + k\frac{\vec{r}}{r}\right]$$
$$= \frac{1}{(2m)^2}[\vec{p}^2, \vec{L}\times\vec{p} - \vec{p}\times\vec{L}] + \frac{k}{2m}\left[\vec{L}\times\vec{p} - \vec{p}\times\vec{L}, \frac{1}{r}\right] + \frac{k}{2m}\left[\vec{p}^2, \frac{\vec{r}}{r}\right],$$

using that x_i commutes with x_j. The first term vanishes as p_i commutes with $\vec{p}^2$, and so does L_i due to 25d.

30c cont)

$$[(\vec{L}\times\vec{p}-\vec{p}\times\vec{L})_i, \tfrac{1}{r}] = \epsilon_{ijk}[L_j p_k - p_j L_k, \tfrac{1}{r}] =$$

$$= \epsilon_{ijk}\{L_j[p_k, \tfrac{1}{r}] + [p_k, \tfrac{1}{r}]L_j\}$$

So we need to compute $[p_k, \frac{1}{r}]$ and $[\vec{p}^2, \frac{x_j}{r}]$.

$$[p_k, x_j] = -i\hbar\delta_{jk} \Rightarrow$$

$$[p_k, r^2] = [p_k, x_j x_j] = -2i\hbar\delta_{jk}x_j = -2i\hbar x_k$$

$$[p_k, f(r)] = f'(r)\frac{\partial\sqrt{r^2}}{\partial r^2}[p_k, r^2] = -\frac{f'(r)}{2r}2i\hbar x_k = -i\hbar\frac{f'(r)}{r}x_k$$

And in particular $[p_k, \frac{1}{r}] = +i\hbar\frac{x_k}{r^3}$.

$$[\vec{p}^2, \tfrac{x_j}{r}] = p_k[p_k, \tfrac{x_j}{r}] + [p_k, \tfrac{x_j}{r}]p_k$$

$$[p_k, \tfrac{x_j}{r}] = -i\hbar\frac{\delta_{jk}}{r} + i\hbar\frac{x_j x_k}{r^3} = -i\hbar\frac{1}{r}(\delta_{jk} - \frac{x_j x_k}{r^2})$$

Hence the 3rd term in $[H, K_j]$ reads:

$$[\vec{p}^2, \tfrac{x_j}{r}] = -i\hbar[p_j\tfrac{1}{r} + \tfrac{1}{r}p_j - p_k\frac{x_k x_j}{r^3} - \frac{x_j x_k}{r^3}p_k]$$

For the 2nd term we find:

$$[(\vec{L}\times\vec{p}-\vec{p}\times\vec{L})_i, \tfrac{1}{r}] = \epsilon_{ijk}\{L_j i\hbar\frac{x_k}{r^3} + i\hbar\frac{x_k}{r^3}L_j\}$$

Insert $L_j = -\epsilon_{jmn}p_m x_n = \epsilon_{jmn}x_m p_n$

$$= -\epsilon_{ijk}\epsilon_{jmn}(i\hbar)(p_m x_n\frac{x_k}{r^3} - \frac{x_k}{r^3}x_m p_n)$$

$$= (\delta_{im}\delta_{kn} - \delta_{in}\delta_{mk})i\hbar(p_m x_n\frac{x_k}{r^3} - \frac{x_k}{r^3}x_m p_n)$$

$$= i\hbar[p_i\tfrac{1}{r} + \tfrac{1}{r}p_i - p_k\frac{x_k x_i}{r^3} - \frac{x_i x_k}{r^3}p_k]$$

Notice that this shows that the 2nd and 3rd terms are equal but opposite! Hence $[H, \vec{K}] = 0$.