3) $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

a) $\sigma_1 \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 \Rightarrow [\sigma_1, \sigma_2] = 2i\sigma_3$

$\sigma_2 \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1 \Rightarrow [\sigma_2, \sigma_3] = 2i\sigma_1$

$\sigma_3 \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \Rightarrow [\sigma_3, \sigma_1] = 2i\sigma_2$

Combining gives: $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$

b) $\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

$\sigma_3^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

$\sigma_1\sigma_2 + \sigma_2\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0$

$\sigma_2\sigma_3 + \sigma_3\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0$

$\sigma_3\sigma_1 + \sigma_1\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0$

Together: $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$.

c) $\sigma_i\sigma_j = \frac{1}{2}\{\sigma_i, \sigma_j\} + \frac{1}{2}[\sigma_i, \sigma_j] = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$

$C = AB = (a_0\mathbb{1} + ia_i\sigma_i)(b_0\mathbb{1} + ib_j\sigma_j)$

$= a_0 b_0 \mathbb{1} + i(b_0 a_i + a_0 b_i)\sigma_i - a_i b_j \underbrace{\sigma_i \sigma_j}_{\delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k}$

$C = (a_0 b_0 - \vec{a}\cdot\vec{b})\mathbb{1} + i(a_0\vec{b} + b_0\vec{a} + \vec{a}\times\vec{b})\cdot\vec{\sigma}$.

d) $\mathrm{tr}\,\mathbb{1} = 2$, $\mathrm{tr}\,\sigma_i = 0$, so that:

$\frac{1}{2}\mathrm{tr}A = \frac{1}{2}\mathrm{tr}(a_0\mathbb{1} + a_i\sigma_i) = \frac{1}{2}\cdot 2a_0 = a_0$

$\frac{1}{2}\mathrm{tr}(\sigma_i A) = \frac{1}{2}\mathrm{tr}[\sigma_i(a_0\mathbb{1} + a_j\sigma_j)] = \frac{a_j}{2}\mathrm{tr}(\sigma_i\sigma_j)$

$= \frac{a_j}{4}\mathrm{tr}(\{\sigma_i, \sigma_j\}) = \frac{a_j}{4}\mathrm{tr}(2\delta_{ij}\mathbb{1}) = a_j\delta_{ij} = a_i$ ✓

e) We use 13b) to compute:

$R = e^{\frac{i}{2}\vec{e}\cdot\vec{\sigma}}\sigma_i e^{-\frac{i}{2}\vec{e}\cdot\vec{\sigma}} = e^{L_{\frac{i}{2}\vec{e}\cdot\vec{\sigma}}}(\sigma_i) = \sum_{n=0} \frac{1}{n!} L^n_{\frac{i}{2}\vec{e}\cdot\vec{\sigma}}(\sigma_i)$

So we determine $L^n_{\frac{i}{2}\vec{e}\cdot\vec{\sigma}}(\sigma_i)$:

$L_{\frac{i}{2}\vec{e}\cdot\vec{\sigma}}[\sigma_i] = \frac{i}{2}e_j[\sigma_j, \sigma_i] = \frac{i}{2}e_j 2i\epsilon_{jik}\sigma_k = \epsilon_{ijk}e_j\sigma_k$

$L^2_{\frac{i}{2}\vec{e}\cdot\vec{\sigma}}[\sigma_i] = \frac{i}{2}e_l[\sigma_l, \epsilon_{ijk}e_j\sigma_k] = \frac{i}{2}e_l e_j\epsilon_{ijk}[\sigma_l, \sigma_k]$

$= \frac{i}{2}e_l e_j\epsilon_{ijk}2i\epsilon_{lkm}\sigma_m = (\epsilon_{ijk}\epsilon_{lmk})e_l e_j\sigma_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})e_l e_j\sigma_m = -e^2\left(\delta_{im} - \frac{e_i e_m}{e^2}\right)\sigma_m$

$= -e^2 P_{ij}\sigma_j$, $P_{ij} = \delta_{ij} - \frac{e_i e_j}{e^2}$,

Note that P is a projector $P^2 = P$, and

$P_{ij}e_j = \left(\delta_{ij} - \frac{e_i e_j}{e^2}\right)e_j = 0$.

31 (cont.)

$$P_{ij} P_{jk} = \left(\delta_{ij} - \frac{e_i e_j}{e^2}\right)\left(\delta_{jk} - \frac{e_j e_k}{e^2}\right)$$

$$= \delta_{ik} - 2\frac{e_i e_k}{e^2} + \frac{e_i e_j e_j e_k}{(e^2)^2} = \delta_{ik} - \frac{e_i e_k}{e^2} = P_{ik}$$

Therefore:

$$L^{2n}_{\frac{1}{2}\vec{e}\cdot\vec{\sigma}}(\sigma_i) = \left[\left(-e^2 P\right)^n\right]_{ij} \sigma_j = (-)^n e^{2n} P_{ij} \sigma_j$$

$$L^{2n+1}_{\frac{1}{2}\vec{e}\cdot\vec{\sigma}}(\sigma_i) = L_{\frac{1}{2}\vec{e}\cdot\vec{\sigma}}\left(L^{2n}_{\frac{1}{2}\vec{e}\cdot\vec{\sigma}}(\sigma_i)\right) = (-)^n e^{2n} P_{ij} L_{\frac{1}{2}\vec{e}\cdot\vec{\sigma}}(\sigma_j)$$

$$= (-)^n e^{2n}\left(\delta_{ij} - \frac{e_i e_j}{e^2}\right) \epsilon_{jkl}\, e_k \sigma_l$$

$$= (-)^n e^{2n+1} \epsilon_{ijk}\, e_j \sigma_k \text{ , because } \epsilon_{jkl}\, e_j e_k = 0.$$

Putting everything together:

$$R = \sum_{n\geq 0} \frac{(-)^n e^{2n}}{(2n)!} P_{ij}\sigma_j + \sum_{n\geq 0} \frac{(-)^n e^{2n+1}}{(2n+1)!} \epsilon_{ijk}\, e_j \sigma_k$$

$$= \left[\cos(e) P_{ij} + \sin(e)\, \epsilon_{ikj}\, e_k\right] \sigma_j \overset{!}{=} R_{ij}\sigma_j$$

Hence:

$$R_{ij} = \cos(e) P_{ij} + \sin(e)\, \epsilon_{ikj}\, e_k$$

32) $\rho = \frac{1}{2}(1+\vec{e}\cdot\vec{\sigma})$, $\rho^2 = \frac{1}{4}(1+2\vec{e}\cdot\vec{\sigma}+(\vec{e}\cdot\vec{\sigma})^2)$

a) $= \frac{1}{4}(1+2\vec{e}\cdot\vec{\sigma}+\vec{e}^2) = \frac{1}{2}(1+\vec{e}\cdot\vec{\sigma}) = \rho$,

because of 31) and $\vec{e}$ a unit vector.

For a pure state $\rho^2=\rho$, hence ρ defines a pure state.

) We want to find $\vec{e}$ such that $|\chi\rangle$ is an eigenfunction of $\frac{\vec{e}\cdot\vec{\sigma}}{2}$: $\frac{\vec{e}\cdot\vec{\sigma}}{2}|\chi\rangle = \pm\frac{1}{2}|\chi\rangle$.

The sign ambiguity arises because $\left(\frac{\vec{e}\cdot\vec{\sigma}}{2}\right)^2 = \frac{1}{4}$, hence $\frac{\vec{e}\cdot\vec{\sigma}}{2}$ has eigenvalues $\pm\frac{1}{2}$.

Write $\frac{1}{2}\vec{e}\cdot\vec{\sigma}$ out explicitly:

$$\vec{e}\cdot\vec{\sigma} = \begin{pmatrix} e_3 & e_1-ie_2 \\ e_1+ie_2 & -e_3 \end{pmatrix}, \quad |\chi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ (1+i)/2 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} \\ (1+i)/2 \end{pmatrix} \frac{1}{=} \vec{e}\cdot\vec{\sigma}\begin{pmatrix} 1/\sqrt{2} \\ (1+i)/2 \end{pmatrix} = \begin{pmatrix} e_3 & e_1-ie_2 \\ e_1+ie_2 & -e_3 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ (1+i)/2 \end{pmatrix}$$

$$= \begin{pmatrix} e_3/\sqrt{2}+e_1/2+e_2/2+i(e_1-e_2)/2 \\ e_1/\sqrt{2}-e_3/2+i(e_2/\sqrt{2}-e_3/2) \end{pmatrix}$$

This gives 4 real equations for 3 real variables:

$\frac{1}{2}e_1+\frac{1}{2}e_2+\frac{1}{\sqrt{2}}e_3 = \frac{1}{\sqrt{2}}$, $e_1-e_2=0$

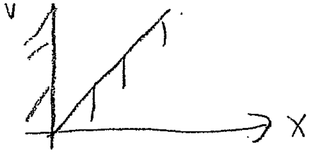
$\frac{1}{\sqrt{2}}e_1-\frac{1}{2}e_3 = \frac{1}{2}$, $\frac{1}{\sqrt{2}}e_2-\frac{1}{2}e_3=\frac{1}{2}$

Hence $e_2=e_1$, which makes the equations on 2nd line the same, hence we need to solve

$e_1+\frac{1}{\sqrt{2}}e_3 = \frac{1}{\sqrt{2}}$, $e_1-\frac{1}{\sqrt{2}}e_3 = \frac{1}{\sqrt{2}} \Rightarrow \begin{matrix} e_3=0 \\ e_1=\frac{1}{\sqrt{2}} \end{matrix}$

Given the sign ambiguity: $\vec{e} = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

33) $V(x) = \begin{cases} \infty & x<0 \\ cx & x>0 \end{cases}, \; c>0$



a) For $x=0$ we get the boundary condition $\psi_\alpha(0)=0$, because for $x<0: V(x)=\infty$. This is solved by the trial function.
For $x\to\infty$ we expect that $\psi(x)$ tends to zero fast enough. The exponential function fulfills this, so $\psi_\alpha(x)$ is an appropriate trial function.
It is ground state because has no zeros.

b) We need to calculate:

$$E(\alpha) = \frac{\langle\psi_\alpha|\hat{H}|\psi_\alpha\rangle}{\langle\psi_\alpha|\psi_\alpha\rangle}$$

and find it's extremum:

$$\langle\psi_\alpha|\psi_\alpha\rangle = \int_0^\infty (x e^{-\alpha x})^2 dx = \int_0^\infty dx\, x^2 e^{-2\alpha x}$$
$$= \frac{2!}{(2\alpha)^2} = \frac{1}{2}\frac{1}{\alpha^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\partial_x^2 - cx\right)$$

$$\hat{H}|\psi_\alpha\rangle = -\frac{\hbar^2}{2m}\left(\partial_x^2 - cx\right)\left(e^{-\alpha x}x\right)$$
$$= -\frac{\hbar^2}{2m}\left[-2\alpha - cx^2 + \alpha^2 x\right]e^{-\alpha x}$$

$$\langle\psi_\alpha|\hat{H}|\psi_\alpha\rangle = -\frac{\hbar^2}{2m}\int_0^\infty dx\left[-2\alpha x + \alpha^2 x^2 - cx^3\right]e^{-2\alpha x}$$
$$= -\frac{\hbar^2}{2m}\left[-2\alpha\frac{1!}{2\alpha} + \alpha^2\frac{2!}{(2\alpha)^2} - c\frac{3!}{(2\alpha)^3}\right]$$
$$= -\frac{\hbar^2}{2m}\left[-1 + \frac{1}{2} - \frac{3}{4}\frac{c}{\alpha^3}\right] = +\frac{\hbar^2}{4m}\left[1 + \frac{3}{2}\frac{c}{\alpha^3}\right]$$

So that:

$$E(\alpha) = \frac{\hbar^2}{2m}\left[\alpha^2 + \frac{3}{2}\frac{c}{\alpha}\right], \quad E'(\alpha) = \frac{\hbar^2}{2m}\left[2\alpha - \frac{3}{2}\frac{c}{\alpha^2}\right]$$

$$E'(\alpha_0) \overset{!}{=} 0 \Rightarrow \alpha_0^3 = \frac{3}{4}c \Rightarrow \alpha_0 = \left(\frac{3}{4}c\right)^{1/3}.$$

Hence the estimate gives:

$$E(\alpha_0) = \frac{\hbar^2}{2m}\left[\alpha_0^2 + \frac{3}{2}\left(\frac{4}{3}\alpha_0^3\right)\frac{1}{\alpha_0}\right] = \frac{3\hbar^2}{2m}\alpha_0^2 = \frac{3\hbar^2}{2m}\left(\frac{3}{4}c\right)^{2/3}$$