34a) For the region $a<x<b$ we have two expressions for the wavefunction; these have to be equal in order that the wavefunction is single-valued. For $x<a$ and $x>b$ the solution is exponentially suppressed. The WKB-connection formulae give inside $a<x<b$ region:

$\frac{2A_1}{\sqrt{k}}\sin\left(\int_a^x k\,dx+\frac{\pi}{4}\right)$ and $\frac{2A_2}{\sqrt{k}}\sin\left(\int_x^b k\,dx+\frac{\pi}{4}\right)$.

For these expressions to be equal:

$$A_2\sin\left(\int_x^b dx\,k(x)+\frac{\pi}{4}\right)=A_2\sin\left(-\int_a^x dx\,k(x)+\int_a^b dx\,k(x)+\frac{\pi}{4}\right)$$

$$=-A_2\sin\left(\int_a^x dx\,k(x)-\int_a^b dx\,k(x)-\frac{\pi}{4}\right)$$

$$=A_2\sin\left(\int_a^x dx\,k(x)-\int_a^b dx\,k(x)-\frac{\pi}{4}+\pi\right)$$

$$\stackrel{!}{=}A_1\sin\left(\int_a^x dx\,k(x)+\frac{\pi}{4}\right)\Rightarrow\int_a^b dx\,k(x)=\left(n+\tfrac{1}{2}\right)\pi,$$

$n\in\mathbb{N}$ because: $\sin(\alpha+\beta)=\cos\alpha\sin\beta+\sin\alpha\cos\beta$

take $\beta=\int_a^x dx\,k(x)+\frac{\pi}{4}$, and $\alpha=-\int_a^b dx\,k(x)+\frac{\pi}{2}$. Hence we only find $\sin\beta$ if $\sin\alpha=0$ which gives the condition. Note that:

$$\frac{A_1}{A_2}=\cos\alpha=\cos n\pi=(-)^n.$$

34b) Bohr–Sommerfeld reads $\oint dx\,p(x)=nh$, where $\oint$ denotes a full cycle. Now $p(x)=k(x)\hbar$, and the relation above is only "half" of a cycle hence the WKB quantization gives:

$$\oint p(x)\,dx=2\hbar\int_a^b dx\,k(x)=2\hbar\left(n+\tfrac{1}{2}\right)\pi=\left(n+\tfrac{1}{2}\right)h$$

Hence the result is the same as BS quantization except that now also a zero-point energie is included.

34c) $V(x)=c|x|$ $\quad 0<V(x)=E\Rightarrow\quad a=-E/c,\ b=E/c$

$$\pi\left(n+\tfrac{1}{2}\right)=\int_a^b dx\,k(x)=2\int_0^b dx\sqrt{2m(E-cx)}/\hbar \qquad y=b-x$$

$$=2\frac{\sqrt{2mc}}{\hbar}\int_0^b dx\sqrt{b-x}=2\frac{\sqrt{2mc}}{\hbar}\int_0^b dy\,y^{\frac{1}{2}}$$

$$=2\frac{\sqrt{2mc}}{\hbar}\frac{2}{3}(b)^{3/2}=\frac{4}{3}\frac{\sqrt{2m}}{\hbar c}E^{3/2}.$$

35a) For $b < x$ the barrier extends to infinity hence only the exponential suppressed solution is allowed. Using the WKB-matching formula give the upper form of the wave-function φ in region $a < x < b$.

On the contrary the barrier at $x = a$ is finite, it ranges from $-a < x < a$. Therefore in that region we may expect an arbitrary linear combination of exponentially decaying and growing solutions. Using the matching conditions this gives a linear combination of sin and cos functions. Because:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \quad \alpha = \int_a^x k\,dx + \frac{\pi}{4}$$

we can always parametrise them with a single sin with an additional angle β; the lower form of φ.

35b) We perform the same rewriting as in 34a to arrive at the identity:

$$C\sin\left(\int_x^b k(x)dx + \frac{\pi}{4}\right) = C\sin\left(\int_a^x k(x)dx - \int_a^b k(x)dx + \frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$\shortparallel$$

$$D\sin\left(\int_a^x k(x)\,dx + \frac{\pi}{4} + \beta\right) \Rightarrow \int_a^b k(x)\,dx = \left(n+\frac{1}{2}\right)\pi - \beta, \; n \in \mathbb{N}$$

As $n \in \mathbb{N}$, we may take $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. To evaluate the integral we note:

$E = V(x)$ gives a, b: $E = A - ca \Rightarrow a = \frac{A-E}{c}$

$E = c\left(b - \frac{A}{c}\right) \Rightarrow b = \frac{A+E}{c}$

Around x_0, $V(x) = c\left|x - \frac{A}{c}\right|$ for $a < x < b$ hence we get

$$\int_a^b k(x)dx = \frac{\sqrt{2m}}{\hbar} \int_{a=\frac{A-E}{c}}^{b=\frac{A+E}{c}} dx \sqrt{E - c\left|x - \frac{A}{c}\right|} =$$

$$= 2\frac{\sqrt{2mc}}{\hbar}\int_{A/c}^{\frac{A+E}{c}} dx \sqrt{\frac{E+A}{c} - x} \overset{y = \frac{E+A}{c} - x}{=} 2\frac{\sqrt{2mc}}{\hbar}\int_0^{E/c} dy\, y^{\frac{1}{2}}$$

$$= \frac{4}{3}\frac{\sqrt{2m}}{c}E^{\frac{3}{2}} \text{ hence: } \frac{4}{3}\frac{\sqrt{2m}}{c}E^{\frac{3}{2}} = \left(n+\frac{1}{2}\right)\pi + \beta.$$

35c) Using the expansion of $\sin(\alpha+\beta)$ given above and the WKB-matching formulae we find that a representation of the wave function in $-a<x<a$ reads:

$$\varphi(x) = \frac{\cos\beta}{\sqrt{k(x)}} e^{-\int_x^a k(x)dx} + \frac{2\sin\beta}{\sqrt{k(x)}} e^{\int_x^a k(x)dx}$$

35 cont.) Since $V(x)=V(-x)$ the wave-function are decomposable in even and odd: $\varphi_\pm(-x)=\pm\varphi(x)$.

To see this reflection properties in the wave-function in $-a<x<a$ we rewrite φ as:

$$\varphi(x)=\frac{\cos\beta}{\sqrt{k(x)}}e^{-\int_0^a k(x)dx}e^{\int_0^x k(x)dx}+\frac{2i\sin\beta}{\sqrt{k(x)}}e^{\int_0^a k(x)dx}e^{-\int_0^x k(x)dx}$$

Since $k(x)=k(-x)$, we see that $\varphi(x)=\pm\varphi(-x)$ requires that:

$$\cos\beta\, e^{-\int_0^a k(x)dx}=\pm 2i\sin\beta\, e^{\int_0^a k(x)dx}$$

$$\tan\beta=\pm\tfrac{i}{2}e^{-2\int_0^a k(x)dx}=\pm\tfrac{i}{2}e^{-\frac{4}{3}\frac{\sqrt{2m}}{\hbar c}(A-E)^{3/2}}$$

because:

$$\int_0^a k(x)dx=\int_0^a \sqrt{2m(A-cx-E)}/\hbar\, dx=\qquad \frac{A-E}{c}-x=y$$

$$=\int_0^{\frac{A-E}{c}}dy\,\frac{\sqrt{2mc}}{\hbar}y^{\frac12}=\frac{2}{3}\frac{\sqrt{2m}}{\hbar c}(A-E)^{\frac32}$$

36a) The Hamiltonian in the presence of a background EM-field reads:

$$H+e\varphi=\frac{1}{2m}\left(\vec p+\frac{e}{c}\vec A\right)^2$$

Hence if $\psi\sim e^{i(Et-\vec p\cdot\vec x)/\hbar}$ is the plain-wave solution without EM-fields, it reads:

$$\psi'\sim e^{i((E+e\varphi)t-(\vec p+\frac{e}{c}\vec A)\cdot\vec x)/\hbar}$$

for constant $\varphi,\vec A$.

36b) $\vec B=\nabla\times\vec A$, $\vec E=-\nabla\varphi-\frac{1}{c}\partial_t\vec A$

As $\vec A$ and φ are constant $\vec B=\vec E=0$.

36c) The gauge transformations read:

$$\varphi'=\varphi-\tfrac{1}{c}\partial_t\chi\quad,\quad \vec A'=\vec A+\nabla\chi$$

We want $\varphi'=\vec A'=0$ for constant $\varphi,\vec A$. Hence we can take

$$\chi=c\varphi t-\vec A\cdot\vec x.$$

36d) Under a gauge transformation the wave function transforms as:

$$\psi\to\psi'=e^{ie\chi/c\hbar}\psi=e^{ie(\varphi t-\frac{1}{c}\vec A\cdot\vec x)/\hbar}e^{i(Et-\vec p\cdot\vec x)/\hbar}$$

Equal to the result in 36a.