37a) $\vec{S} = \frac{1}{2}\vec{\sigma}$; $|E,+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|E,-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ is a basis of eigenstates of $S_z = \frac{1}{2}\sigma_3$, and $H_0|E,\pm\rangle = E|E,\pm\rangle$.

The first order perturbation of the energie levels is given by:

$$E^{(1)}_{\pm} = \langle E,\pm|W|E,\pm\rangle,$$

where:

$$W = -\vec{\mu}\cdot\vec{B} = -2\mu_B\frac{1}{\hbar}\vec{S}\cdot\vec{B} = -2\frac{e\hbar}{2m}\frac{1}{\hbar}\frac{1}{2}\sigma_3 B = -\frac{eB}{2m}\sigma_3$$

So that:

$$E^{(1)}_{\pm} = \langle E,\pm| -\frac{eB}{2m}\sigma_3|E,\pm\rangle = \mp\frac{eB}{2m}$$

37b) The eigenvectors are $|E,\pm\rangle$.

38a) The Hamiltonian H_0 is the sum of two Hamiltonians of identical harmonic oscillators ($i=1,2$):

$$H_i = \frac{1}{2m}P_i^2 + \frac{1}{2}m\omega^2 x_i^2 = \hbar\omega(N_i + \tfrac{1}{2}), \quad N_i = a_i^\dagger a_i$$

where:

$$x_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i^\dagger + a_i); \quad P_i = i\sqrt{\frac{m\hbar\omega}{2}}(a_i^\dagger - a_i)$$

Hence the energie spectrum is determined by $H_0 = H_1 + H_2$, i.e. the quantum numbers n_1, n_2: $E^{(0)} = \hbar\omega(n_1+n_2+1)$, $|n_1,n_2\rangle$ states.

$E^{(0)}/\hbar\omega$	(n_1,n_2)	# states
1	$(0,0)$	1
2	$(1,0);(0,1)$	2
3	$(2,0);(1,1);(0,2)$	3

38b) $N_i|n_1,n_2\rangle = n_i|n_1,n_2\rangle$

$a_1|n_1,n_2\rangle = \sqrt{n_1}|n_1-1,n_2\rangle$, $a_1^\dagger|n_1,n_2\rangle = \sqrt{n_1+1}|n_1+1,n_2\rangle$

$a_2|n_1,n_2\rangle = \sqrt{n_2}|n_1,n_2-1\rangle$, $a_2^\dagger|n_1,n_2\rangle = \sqrt{n_2+1}|n_1,n_2+1\rangle$

The general expressions for 1st and 2nd order perturbations of energie levels read:

$$E^{(1)}_{n_1,n_2} = \langle n_1,n_2|W|n_1,n_2\rangle$$

$$E^{(2)}_{n_1,n_2} = \sum_{(n_1',n_2')\neq(n_1,n_2)} \frac{|\langle n_1',n_2'|W|n_1,n_2\rangle|^2}{E^{(0)}_{n_1',n_2'} - E^{(0)}_{n_1,n_2}}$$

The interaction can be written as:

$$W = \gamma\frac{m^2\omega^3}{\hbar^2}x_1^2x_2^2 = \gamma\frac{m^2\omega^3}{\hbar^2}\left(\frac{\hbar}{2m\omega}\right)^2(a_1^\dagger+a_1)^2(a_2^\dagger+a_2)^2$$

$$= \frac{\gamma}{4}\hbar\omega(a_1^{\dagger 2}+a_1^2+2N_1+1)(a_2^{\dagger 2}+a_2^2+2N_2+1)$$

38b cont) For the ground state $|0,0\rangle$ this amounts to:

$$E^{(1)}_{0,0} = \langle 0,0|W|0,0\rangle = \frac{\gamma}{4}\hbar\omega$$

$$E^{(2)}_{0,0} = -\sum_{\substack{(n_1',n_2')=(2,0),\\ (0,2),(2,2)}} \frac{|\langle n_1',n_2'|W|0,0\rangle|^2}{E^{(0)}_{n_1',n_2'} - E^{(0)}_{0,0}}$$

To evaluate this note that:

$$E^{(0)}_{n_1',n_2'} - E^{(0)}_{0,0} = \hbar\omega(n_1'+n_2')$$

$$\langle 2,0|W|0,0\rangle = \frac{\gamma}{4}\hbar\omega\langle 2,0|a_1^{\dagger 2}|0,0\rangle = \frac{\gamma}{4}\hbar\omega\sqrt{2} = \frac{\gamma}{2\sqrt{2}}\hbar\omega$$

$$\langle 0,2|W|0,0\rangle = \frac{\gamma}{4}\hbar\omega\langle 0,2|a_2^{\dagger 2}|0,0\rangle = \frac{\gamma}{4}\hbar\omega\sqrt{2} = \frac{\gamma}{2\sqrt{2}}\hbar\omega$$

$$\langle 2,2|W|0,0\rangle = \frac{\gamma}{4}\hbar\omega\langle 2,2|a_1^{\dagger 2}a_2^{\dagger 2}|0,0\rangle = \frac{\gamma}{4}\hbar\omega\sqrt{2}^2 = \frac{1}{2}\gamma\hbar\omega$$

Putting things together:

$$E^{(2)}_{0,0} = -2\cdot\frac{\left(\frac{\gamma}{2\sqrt{2}}\hbar\omega\right)^2}{2\hbar\omega} - \frac{\left(\frac{1}{2}\gamma\hbar\omega\right)^2}{4\hbar\omega} = -\gamma^2\hbar\omega\left(\frac{1}{8}+\frac{1}{16}\right) = -\frac{3}{16}\gamma^2\hbar\omega.$$

38c)

Because at this level the states $|1,0\rangle, |0,1\rangle$ are degenerate, the corrections to 1st order of the energy levels is determined by diagonalizing the matrix

$$\hat{W} = \begin{pmatrix} \langle 1,0|W|1,0\rangle & \langle 1,0|W|0,1\rangle \\ \langle 0,1|W|1,0\rangle & \langle 0,1|W|0,1\rangle \end{pmatrix}.$$

$$= \frac{\gamma}{4}\hbar\omega\begin{pmatrix} 2+1 & 0 \\ 0 & 2+1 \end{pmatrix} = \frac{3}{4}\gamma\hbar\omega\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\hat{W}$ is already diagonal, the degeneration remains to first order.

38d) At this level the states $|2,0\rangle, |0,2\rangle$ $|1,1\rangle$ are degenerate: The corresponding matrix $\hat{W}$ reads:

$$\hat{W} = \begin{pmatrix} \langle 2,0|W|2,0\rangle & \langle 2,0|W|0,2\rangle & \langle 2,0|W|1,1\rangle \\ \langle 0,2|W|2,0\rangle & \langle 0,2|W|0,2\rangle & \langle 0,2|W|1,1\rangle \\ \langle 1,1|W|2,0\rangle & \langle 1,1|W|0,2\rangle & \langle 1,1|W|1,1\rangle \end{pmatrix}$$

We have

$$\langle 2,0|W|2,0\rangle = \langle 0,2|W|0,2\rangle = \frac{\gamma}{4}\hbar\omega(2\cdot 2+1) = \frac{5}{4}\gamma\hbar\omega$$

$$\langle 1,1|W|1,1\rangle = \frac{\gamma}{4}\hbar\omega(2\cdot 1+1)^2 = \frac{9}{4}\gamma\hbar\omega$$

$$\langle 2,0|W|0,2\rangle = \langle 0,2|W|2,0\rangle = \frac{1}{4}\gamma\hbar\omega(\sqrt{2})^2 = \frac{1}{2}\gamma\hbar\omega$$

So that

$$\hat{W} = \frac{1}{4}\gamma\hbar\omega\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

38d cont.) We see that the state $|1,1\rangle$ does not rise and has energiecorrection $E^{(1)}_{1,1} = \frac{9}{4}\gamma\hbar\omega$.

The eigenvalues for the other two states are determined as:

$$0 \overset{!}{=} \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 4 \Rightarrow \lambda = 5 \pm 2 = \begin{cases} 7 \\ 2 \end{cases}$$

$$\lambda=7: \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0, \quad \lambda = 3: \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence the normalised states:

$$|(2,0),\pm\rangle = \frac{1}{\sqrt{2}}\left(|2,0\rangle \pm |0,2\rangle\right)$$ have energie corrections at 1st order of:

$$E_{(2,0),+} = \frac{3}{4}\gamma\hbar\omega, \quad E_{(2,0),-} = \frac{7}{4}\gamma\hbar\omega.$$

The degeneracy is completely removed.

39a)
The transition probability to go from state $|n\rangle$ at t_0 to state $|m \neq n\rangle$ at t reads

$$P_{n\to m}(t,t_0) = \frac{1}{\hbar^2}\left|\int_{t_0}^{t} d\tau\, e^{\frac{i}{\hbar}(E_m - E_n)\tau} \langle m|W|n\rangle\right|^2$$

with W the interaction. In the present case:

$$W = qE(t)x = q\sqrt{\frac{\hbar}{2m\omega}}\,\frac{A}{\sqrt{\pi}\tau_0} e^{-t^2/\tau_0^2}(a + a^\dagger)$$

$$P_{0\to n}(-\infty,\infty) = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} d\tau\, e^{in\omega\tau - \tau^2/\tau_0^2}\, q\sqrt{\frac{\hbar}{2m\omega}}\,\frac{A}{\sqrt{\pi}\tau_0}\langle n|a+a^\dagger|0\rangle\right|^2$$

$$= \frac{q^2A^2}{2\pi m\omega\tau_0^2}\left|\int_{-\infty}^{\infty} d\tau\, e^{in\omega\tau - \tau^2/\tau_0^2}\right|^2 |\langle n|1\rangle|^2$$

Note that $\langle n|1\rangle = \delta_{n1}$, and

$$\int_{-\infty}^{\infty} d\tau\, e^{in\omega\tau - \tau^2/\tau_0^2} \overset{\tau = \tau_0 y}{=} \tau_0 \int_{-\infty}^{\infty} e^{in\omega\tau_0 y - y^2}$$

$$= \tau_0 \int_{-\infty}^{\infty} dy\, e^{-(y - \frac{i}{2}n\omega\tau_0)^2} e^{-\frac{1}{4}n^2\omega^2\tau_0^2} = \sqrt{\pi}\,\tau_0\, e^{-\frac{1}{4}n^2\omega^2\tau_0^2}$$

Hence:

$$P_{0\to n\neq 0}(-\infty,\infty) = \frac{q^2A^2}{2\pi m\omega\tau_0^2}\,\pi\tau_0^2 e^{-\frac{1}{2}n^2\omega^2\tau_0^2}\delta_{n1}$$

$$= \frac{q^2A^2}{2m\omega} e^{-\frac{1}{2}\omega^2\tau_0^2}\delta_{n1}$$

$$P_{0\to 0}(-\infty,\infty) = 1 - \sum_{n>0} P_{0\to n\neq 0}(-\infty,\infty) = 1 - \frac{A^2q^2}{2m\omega} e^{-\frac{1}{2}\omega^2\tau_0^2}$$

39b) The total transition rate $0 \to n\neq 0$ should be small, i.e.

$$\frac{q^2A^2}{2m\omega} e^{-\frac{1}{2}\omega^2\tau_0^2} \ll 1.$$