19a) $H(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$, $V(x) = \frac{\hbar^2}{2m} D \sum_n \delta(x+an)$

$T_{pa}\psi(x) = \psi(x+pa)$ is the translation operator, $p \in \mathbb{Z}$.
Because $T_{pa}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}$ and $T_{pa}(V(x)) = V(x)$, it follows that $[T_{pa}, H(x)] = 0$. Hence we can find a basis of simultaneous eigenvalues of $H$ and $T_{pa}$. Since translation commute and are additive, we have:

$$T_{p_1 a} T_{p_2 a} = T_{p_2 a} T_{p_1 a} = T_{(p_1+p_2)a}$$

Hence there is a basis of eigenstates of $H$ and $T_{pa}$, $p \in \mathbb{Z}$. Let $\psi(x)$ be an element of this basis, then $T_{pa}\psi(x) = c_p \psi(x)$. Using the additivity of translation gives:

$$c_{p_1+p_2}\psi(x) = T_{(p_1+p_2)a}\psi(x) = T_{p_1 a} T_{p_2 a}\psi(x) =$$
$$= c_{p_2} T_{p_1 a}\psi(x) = c_{p_1} c_{p_2}\psi(x) \Rightarrow c_{p_1+p_2} = c_{p_1} c_{p_2}$$

Furthermore, since $T_{pa} = e^{ipa\frac{\partial}{\partial x}}$ it follows that $T_{pa}^{*} = T_{-pa}$, hence $c_p^{*} = c_{-p}$. Combining implies that $c_p = e^{iKpa}$, for some constant $K$, which may be chosen inside the range $-\frac{\pi}{a} < K \leq \frac{\pi}{a}$.

19b) Split $\mathbb{R} = \bigcup_n B_n$, $B_n = \{x \mid na < x < (n+1)a\}$
In each $B_n$, $V(x) = 0$, hence $\psi(x) = A_n e^{ikx} + B_n e^{-ikx}$.
Take $x \in B_0$ then by the Bloch theorem:

$$\psi(x+an) = e^{iKna}\psi(x) = e^{iKan}(A_0 e^{ikx} + B_0 e^{-ikx})$$
$$\shortparallel$$
$$A_n e^{ik(x+na)} + B_n e^{-ik(x+an)} \Rightarrow$$
$$A_n = e^{ni(K-k)a} A_0, \quad B_n = e^{ni(K+k)a} B_0.$$

Next we can use the boundary conditions at, say $x=0$, that the $\delta$-function potential implies:

$$\psi(\varepsilon) = \psi(-\varepsilon), \ \tfrac{1}{2}\left(\psi'(\varepsilon) - \psi'(-\varepsilon)\right) = D\psi(0)$$

Inserting the solutions in $B_0$ and $B_{-1}$ gives:

$$0 = A_0 + B_0 - A_{-1} - B_{-1} = A_0 + B_0 - e^{-i(K-k)a}A_0 - e^{-i(K+k)a}B_0$$

$$0 = \frac{ik}{2D}\left(A_0 - B_0 - (A_{-1} + B_{-1})\right) - A_0 - B_0 =$$
$$= \frac{ik}{2D}\left(A_0 - B_0 - e^{-i(K-k)a}A_0 + e^{-i(K+k)a}B_0\right) - A_0 - B_0.$$

Matrix form:

$$\begin{pmatrix} 1 - e^{-i(K-k)a} & 1 - e^{-i(K+k)a} \\ 1 - \frac{ik}{2D}(1 - e^{-i(K-k)a}) & 1 + \frac{ik}{2D}(1 - e^{-i(K+k)a}) \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = 0$$

9b cont.) Non-trivial solutions when the determinant of this matrix vanishes:

$$\left(1-e^{-i(K-k)a}\right)\left(1+\frac{ik}{2D}\left(1-e^{-i(K+k)a}\right)\right)+$$

$$-\left(1-e^{-i(K+k)a}\right)\left(1-\frac{ik}{2D}\left(1-e^{-i(K-k)a}\right)\right)\overset{!}{=}0$$

$$=\cancel{1}+\frac{ik}{2D}\left(1-e^{-i(K+k)a}\right)-\cancel{1}+\frac{ik}{2D}\left(1-e^{-i(K-k)a}\right)+$$

$$-e^{-i(K-k)a}-\frac{ik}{2D}\left(e^{-i(K-k)a}-e^{-2iKa}\right)+e^{-i(K+k)a}-\frac{ik}{2D}\left(e^{-i(K+k)a}-e^{-2iKa}\right)$$

Multiply by $\frac{D}{ik}e^{iKa}$ gives:

$$-e^{iKa}+e^{-iKa}-e^{ika}-e^{-ika}-\frac{D}{ik}\left(e^{ika}-e^{-ika}\right)=0$$

or:

$$\cos Ka=\cos ka+\frac{D}{k}\sin ka.$$

Since $|\cos Ka|\leq 1$ and there are values for $k$, i.e. $E$, for which $|\cos ka+\frac{D}{k}\sin ka|=\sqrt{1+D^2/k^2}\,|\cos(ka-\varphi)|$, $\varphi=\arctan D/k$, is larger than 1. Such value for $E$ are not solution to this eigenvalue problem.

20a) $E_i\psi_i(x)=\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi_i(x)$

Therefore $W(\psi_1,\psi_2)'=\frac{\hbar^2}{2m}\left(\psi_1\psi_2'-\psi_1'\psi_2\right)'=$

$=\frac{\hbar^2}{2m}\left(\psi_1\psi_2''-\psi_1''\psi_2\right)=\psi_1(V-E_2)\psi_2-(V-E_1)\psi_1\psi_2=$

$=(E_1-E_2)\psi_1\psi_2$. Integrating between a and b gives

$$W(\psi_1,\psi_2)\Big|_a^b=(E_1-E_2)\int_a^b dx\,\psi_1\psi_2.$$

20b) Denote a and b two consecutive zeros of $\psi_1$, then the equation above reduces to:

$$-\frac{\hbar^2}{2m}\psi_1'\psi_2\Big|_a^b=(E_1-E_2)\int_a^b dx\,\psi_1\psi_2.$$

Because the zeros a, b are consecutive, $\psi_1$ does not change sign for $x\in\,]a,b[$. Say $\psi(x)>0$ there, then $\psi'(a)>0$ and $\psi'(b)<0$. Hence the R.H.S. has opposite sign to $\psi_2$, while the L.H.S. the same sign. If $\psi_2$ does not change sign, this equation is therefore never satisfied. Hence $\psi_2$ changes sign, i.e. has zero in $]a,b[$.

20c) By repeating this argument we see that states larger energies have zero in between those of lesser energies, hence have more zero. Hence for $\{\varphi_n\}$ with $\varphi_n$ has $n-1$ zeros, their energies are ordered as $E_1 < E_2 < \ldots < 0$.

21a) Define $\psi_{E,\pm}(x) = \frac{1}{2}\left(\psi_E(x) \pm \psi_E(-x)\right)$.

Since $V(x) = V(-x)$ and $\left(\frac{d}{d(-x)}\right)^2 = \left(\frac{d}{dx}\right)^2$ it follows that if $\psi_E(x)$ is a solution of the Schrödinger-Eq. so is $\psi_E(-x)$, and any linear combination as well.

$$\psi_{E,-}(0) = \frac{1}{2}\left(\psi_E(0) - \psi_E(-0)\right) = 0 \quad \checkmark$$

21b) Because, $U(x) = \infty$ for $x < 0$, it follows that $\psi(x) = 0$, for $x < 0$, so in particular by continuity: $\psi(0) = 0$. For $x \geq 0$, $U(x) = V(x)$ hence the Schrödinger-Eq.'s are identical, hence so are the solutions. Since the solutions are spanned as: $\psi(x) = a_+\psi_{E,+}(x) + a_-\psi_{E,-}(x)$, $x \geq 0$, we only have to impose the boundary condition: $\psi(0) = 0$. It follows that $a_+ = 0$. So the eigenstates of $U(x)$ are the odd eigenstate $\psi_{E,-}$ of $V(x)$.

21c) The solution of the harmonic oscillator are $\varphi_n(x) \sim H_n(z)\, e^{-z^2/2}$, $z = \sqrt{\frac{m\omega}{\hbar}}\, x$.

Since the Hermite polynomials satisfy:
$H_n(-z) = (-)^n H_n(z)$, we see that the $\varphi_n(x)$ with $n$ odd are odd functions. Set $n = 2p+1$, $p \in \mathbb{N}$ then

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right) = \hbar\omega\left(2p+1+\tfrac{1}{2}\right) = \hbar\omega\left(2p + \tfrac{3}{2}\right).$$