22a) <u>n non-negative:</u>

$n = \langle n|\hat{N}|n\rangle = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = \| \hat{a}|n\rangle \|^2$,

because the norm is non-negative, so is n.

<u>n is integral:</u>

Assume n is not integral, then we can write $n = m + x$, with $0 < x < 1$, $m \in \mathbb{N}$.

Since $[\hat{N}, \hat{a}] = -\hat{a}$, it follows that the state $\hat{a}^{m+1}|n\rangle$ has eigenvalue $x-1$:

$\hat{N}\hat{a}^{m+1}|n\rangle = \hat{a}^{m+1}(-(m+1) + \hat{N})|n\rangle =$

$(-(m+1) + m + x)\hat{a}^{m+1}|n\rangle = (x-1)\hat{a}^{m+1}|n\rangle$.

But then it eigenvalue is negative: $x-1<0$, which is in contradiction with the conclusion above.

b) $\| \hat{a}|0\rangle \|^2 = \langle 0|\hat{a}^\dagger \hat{a}|0\rangle = \langle 0|\hat{N}|0\rangle = 0$.

From $|n\rangle$, we obtain a state $\hat{a}^{n+p}|n\rangle$ that has $\hat{N}\hat{a}^{n+p}|n\rangle = -p\hat{a}^{n+p}|n\rangle$, $p \geq 1$. It is zero because $\hat{a}^{n+p}|n\rangle \sim \hat{a}^p|0\rangle$, $\hat{a}^p|0\rangle = \hat{a}^{p-1}\hat{a}|0\rangle = 0$.

c) <u>n unbounded:</u>

Suppose, there is a N_{max}: $0 \leq n \leq N_{max}$. Then $\hat{a}^\dagger|N_{max}\rangle = 0$, but:

$\| \hat{a}^\dagger|N_{max}\rangle \|^2 = \langle N_{max}|\hat{a}\hat{a}^\dagger|N_{max}\rangle =$

$= \langle N_{max}|(1 + \hat{N})|N_{max}\rangle = 1 + N_{max} \geq 1 \neq 0$,

using that $|N_{max}\rangle \neq 0$ and normalized. Hence $\hat{a}^\dagger|N_{max}\rangle$ is not a zero-state.

<u>non-degenerate:</u>

Suppose that the spectrum is degenerate. Then there exists some operator $\hat{F}$ that commutes with $\hat{N}$. This operator can be expanded as:

$\hat{F} = \sum_{n,m\geq 0} F_{nm}\hat{a}^{\dagger n}\hat{a}^m$.

(Using $[\hat{a}, \hat{a}^\dagger] = 1$ we can always bring it in this form.) Since $\hat{F}$ commutes with $\hat{N}$ we have:

$0 = [\hat{N}, \hat{F}] = \sum_{n,m\geq 0} F_{nm}[\hat{N}, \hat{a}^{\dagger n}\hat{a}^m] = \sum_{n,m\geq 0}(n-m)F_{nm}\hat{a}^{\dagger n}\hat{a}^m$.

This implies for $n \neq m$: $F_{nm} = 0$, hence $\hat{F}$ is a function $f(\hat{N})$ only, so that $\hat{F}|n\rangle = f(n)|n\rangle \sim |n\rangle$, not independent.

23b) $\sum_{n\geq 0} \frac{s^n}{n!} H_n(z) := e^{-s^2+2zs}$

$\sum_{n\geq 0} \frac{s^n}{n!} \frac{d}{dz} H_n(z) = \frac{d}{dz}\left(e^{-s^2+2zs}\right) = 2s\, e^{-s^2+2zs}$

$= 2\sum_{n\geq 0} \frac{(n+1)s^{n+1}}{(n+1)n!} H_n(z) \overset{n\to n-1}{=} \sum_{n\geq 0} \frac{s^n}{n!}\left(2n H_{n-1}(z)\right) \Rightarrow \frac{d}{dz}H_n = 2n H_{n-1}$

(Note: $n=0$ gives no contribution in last sum.)

Apply $\frac{d}{ds}$ to the generating function:

$\frac{d}{ds} \sum_{n\geq 0} \frac{s^n}{n!} H_n(z) = \sum_{n\geq 1} \frac{n s^{n-1}}{n!} H_n(z) \overset{n\to n+1}{=} \sum_{n\geq 0} \frac{s^n}{n!} H_{n+1}(z)$

$\|$

$\frac{d}{ds} e^{-s^2+2zs} = (-2s+2z)e^{-s^2+2zs} = \sum_{n\geq 0} \frac{s^n}{n!}\left(-2n H_{n-1}(z) + 2z H_n(z)\right)$

$\Rightarrow H_{n+1}(z) = -2n H_{n-1}(z) + 2z H_n(z).$

23a) Note that $s \frac{d}{ds} \sum_{n\geq 0} \frac{s^n}{n!} H_n(z) = \sum_{n\geq 0} \frac{s^n}{n!} n H_n(z).$

Therefore: $\sum_{n\geq 0} \frac{s^n}{n!}\left[\frac{d^2}{dz^2} - 2z\frac{d}{dz} + 2n\right] H_n(z) =$

$= \left[\frac{d^2}{dz^2} - 2z\frac{d}{dz} + 2s\frac{d}{ds}\right] \sum_{n\geq 0} \frac{s^n}{n!} H_n(z) = \left[\frac{d^2}{dz^2} - 2z\frac{d}{dz} + 2s\frac{d}{ds}\right] e^{-s^2+2zs}$

$= \left[(2s)^2 - 2z(2s) + 2s(-2s+2z)\right] e^{-s^2+2zs} = 0$, hence:

$\left[\frac{d^2}{dz^2} - 2z\frac{d}{dz} + 2n\right] H_n(z) = 0.$

23c) $\sum_{n,m\geq 0} \frac{s^n t^m}{n!\,m!} \int_{-\infty}^{\infty} dz\, H_n(z) H_m(z) e^{-z^2} =$

$= \int_{-\infty}^{\infty} dz\, e^{-z^2} e^{-s^2+2zs} e^{-t^2+2zt} = \int_{-\infty}^{\infty} dz\, e^{-(z-s-t)^2+2st}$

$= \sqrt{\pi} \sum_{p\geq 0} \frac{2^p s^p t^p}{p!} \Rightarrow \int_{-\infty}^{\infty} dz\, e^{-z^2} H_n(z) H_m(z) = \sqrt{\pi}\, 2^n n!\, \delta_{n,m}.$

23d) $\sum_{n\geq 0} \frac{s^n}{n!} e^{z^2}\left(-\frac{d}{dz}\right)^n e^{-z^2} = e^{z^2} \sum_{n\geq 0} \frac{1}{n!}\left(-s\frac{d}{dz}\right)^n e^{-z^2} =$

$= e^{z^2} e^{-s\frac{d}{dz}}\left(e^{-z^2}\right) = e^{z^2} e^{-(z-s)^2} = e^{-s^2+2zs},$

using that $e^{-s\frac{d}{dz}} \varphi(z) = \varphi(z-s).$

24a) $\psi(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-(z-z_0)^2/2}$. Write in e^{-s^2+2sz} form:

$$\psi(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-(z_0/2)^2+2(z_0/2)z} e^{-z_0^2/4} e^{-z^2/2}$$

$$= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \sum_{n\geq 0} \frac{1}{n!}\left(\frac{z_0}{2}\right)^n e^{-z_0^2/4} H_n(z) e^{-z^2/2}$$

24b) We want to expansion

$$\psi(x) = \sum_{n\geq 0} A_n \varphi_n(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \sum_{n\geq 0} \frac{A_n}{\sqrt{2^n n!}} H_n(z) e^{-z^2/2}$$

$$\Rightarrow A_n = \frac{1}{\sqrt{n!}}\left(\frac{z_0}{\sqrt{2}}\right)^n e^{-z_0^2/4}$$

As φ_n are normalized eigenstates of $\hat{N}$, the probability to find $\hat{N} = n$ is given by:

$$P_n = |A_n| = \frac{1}{n!}\left(\frac{z_0^2}{2}\right)^n e^{-z_0^2/2}$$

24c) $\sum_{n\geq 0} P_n = \sum_{n\geq 0} \frac{1}{n!}\left(\frac{z_0^2}{2}\right)^n e^{-z_0^2/2} = e^{z_0^2/2} e^{-z_0^2/2} = 1$

Differentiate this equation w.r.t. z_0:

$$0 = z_0 \frac{d}{dz_0}\left[\sum_{n\geq 0} \frac{1}{n!}\left(\frac{z_0^2}{2}\right)^n e^{-z_0^2/2}\right] = \sum_{n\geq 0} \frac{2n - z_0^2}{n!}\left(\frac{z_0^2}{2}\right)^n e^{-z_0^2/2}$$

$$= 2\sum_{n\geq 0} n \frac{1}{n!}\left(\frac{z_0^2}{2}\right)^n e^{-z_0^2/2} - z_0^2 = 2\sum_{n\geq 0} n P_n - z_0^2$$

$$\Rightarrow \langle \hat{N} \rangle = \sum_{n\geq 0} n P_n = z_0^2/2.$$