25a) $\epsilon_{ijk}\epsilon_{klm}$ is anti-symmetric in (i,j) and (l,m). Because of complete anti-symmetry is either:

$(i,j)=(l,m)$: $\epsilon_{ijk}\epsilon_{kij}=(\epsilon_{ijk})^2=1$

or $(i,j)=(m,l)$: $\epsilon_{ijk}\epsilon_{kji}=\epsilon_{ijk}\epsilon_{jik}=-(\epsilon_{ijk})^2=-1$

This is precisely what $\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$.

25b) $[L_i,x_j]=\epsilon_{ikl}[x_k p_l,x_j]=\epsilon_{ikl}x_k[p_l,x_j]$
$=-i\hbar\,\delta_{lj}\epsilon_{ikl}x_k=-i\hbar\,\epsilon_{ikj}x_k=i\hbar\,\epsilon_{ijk}x_k$

$[L_i,p_j]=\epsilon_{ikl}[x_k p_l,p_j]=\epsilon_{ikl}[x_k,p_j]p_l$
$=i\hbar\,\delta_{kj}\epsilon_{ikl}p_l=i\hbar\,\epsilon_{ijk}p_k$.

25c) $[L_i,L_j]=\epsilon_{jkl}[L_i,x_k p_l]=\epsilon_{jkl}([L_i,x_k]p_l+x_k[L_i,p_l])=i\hbar(\epsilon_{jkl}\epsilon_{ikm}x_m p_l+\epsilon_{jkl}\epsilon_{ilm}x_k p_m)$
$=i\hbar(\delta_{ji}\delta_{lm}-\delta_{jm}\delta_{li})x_m p_l+i\hbar(-\delta_{ji}\delta_{km}+\delta_{jm}\delta_{ki})x_k p_m$
$=i\hbar(\delta_{ij}\delta_{lk}-\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl})x_k p_l$
$=i\hbar\,\epsilon_{ijm}\epsilon_{mkl}x_k p_l=i\hbar\,\epsilon_{ijm}L_m$.

25d) $[L_i,\vec{A}^2]=[L_i,A_kA_k]=[L_i,A_k]A_k+A_k[L_i,A_k]=$
$=i\hbar\,\epsilon_{ikm}A_mA_k+i\hbar\,\epsilon_{ikm}A_kA_m=i\hbar(\epsilon_{ikm}+\epsilon_{imk})A_mA_k$
$=0$.

$\vec{A}^2$ is a scalar, i.e. does not transform under rotations. Since L_i generate rotations: $[L_i,\vec{A}^2]=0$

26a) Insert the parity in angular variables:

$x\to x'=r\sin(\pi-\theta)\cos(\varphi+\pi)=(-)^2 r\sin(-\theta)\cos\varphi=-r\sin\theta\cos\varphi=-x$
$y\to y'=r\sin(\pi-\theta)\sin(\varphi+\pi)=(-)r\sin(-\theta)\sin\varphi=-r\sin\theta\sin\varphi=-y$
$z\to z'=r\cos(\pi-\theta)=-r\cos(-\theta)=-r\cos\theta=-z$ ✓

26b)
$$\partial_{r,\theta,\varphi}=\frac{\partial x}{\partial (r,\theta,\varphi)}\partial_x+\frac{\partial y}{\partial (r,\theta,\varphi)}\partial_y+\frac{\partial z}{\partial (r,\theta,\varphi)}\partial_z \Rightarrow$$

$$\begin{pmatrix} r\partial_r\\ \partial_\theta\\ \partial_\varphi\end{pmatrix}=\begin{pmatrix} r\sin\theta\cos\varphi & r\sin\theta\sin\varphi & r\cos\theta\\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta\\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}=rM\begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}$$

$$M^{-1}=\begin{pmatrix}\sin\theta\cos\varphi & \cos\theta\cos\varphi & -\frac{1}{\sin\theta}\sin\varphi\\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \frac{1}{\sin\theta}\cos\varphi\\ \cos\theta & -\sin\theta & 0\end{pmatrix}$$

2.6 contd)

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \frac{1}{r}\begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & \frac{-1}{\sin\theta}\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \frac{1}{\sin\theta}\cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} r\partial_r \\ \partial_\theta \\ \partial_\varphi \end{pmatrix}$$

$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

Substitute this in the component expressions gives:

$L_z = \frac{\hbar}{i}(x\partial_y - y\partial_x) =$

$= \frac{\hbar}{i}\sin\theta\cos\varphi(\sin\theta \sin\varphi r\partial_r + \cos\theta\sin\varphi\partial_\theta + \frac{1}{\sin\theta}\cos\varphi\partial_\varphi) +$

$- \frac{\hbar}{i}\sin\theta\sin\varphi(\sin\theta\cos\varphi r\partial_r + \cos\theta\cos\varphi\partial_\theta - \frac{1}{\sin\theta}\sin\varphi\partial_\varphi) = \frac{\hbar}{i}\partial_\varphi$

$L_x = \frac{\hbar}{i}(y\partial_z - z\partial_y) =$

$= \frac{\hbar}{i}\sin\theta\sin\varphi(\cos\theta r\partial_r - \sin\theta\partial_\theta) +$

$- \frac{\hbar}{i}\cos\theta(\sin\theta\sin\varphi r\partial_r + \cos\theta\sin\varphi\partial_\theta + \frac{1}{\sin\theta}\cos\varphi\partial_\varphi)$

$= -\frac{\hbar}{i}(\sin\varphi\,\partial_\theta + \cot\theta\cos\varphi\,\partial_\varphi)$

$L_y = \frac{\hbar}{i}(z\partial_x - x\partial_z)$

$= \frac{\hbar}{i}\cos\theta(\sin\theta\cos\varphi r\partial_r + \cos\theta\cos\varphi\partial_\theta - \frac{1}{\sin\theta}\sin\varphi\partial_\varphi) +$

$- \frac{\hbar}{i}\sin\theta\cos\varphi(\cos\theta r\partial_r - \sin\theta\partial_\theta)$

$= \frac{\hbar}{i}(\cos\varphi\,\partial_\theta - \cot\theta\sin\varphi\,\partial_\varphi)$

$L_\pm = L_x \pm iL_y = \pm i(L_y \mp iL_x)$

$= \pm\hbar[(\cos\varphi \pm i\sin\varphi)\partial_\theta - (\sin\varphi \mp i\cos\varphi)\cot\theta\,\partial_\varphi]$

$e^{\pm i\varphi} = \cos\varphi \pm i\sin\varphi$, $ie^{\pm i\varphi} = \mp\sin\varphi + i\cos\varphi = \mp(\sin\varphi \mp i\cos\varphi)$

so that:

$L_\pm = \pm\hbar[e^{\pm i\varphi}\partial_\theta \pm ie^{\pm i\varphi}\cot\theta\,\partial_\varphi] = \pm\hbar e^{\pm i\varphi}[\partial_\theta \pm i\cot\theta\,\partial_\varphi]$

2.6c) From $\frac{\hbar}{i}\partial_\varphi Y_\ell^{\pm\ell} = L_z Y_\ell^{\pm\ell} = \pm\hbar\ell Y_\ell^{\pm\ell}$ it follows that $Y_\ell^{\pm\ell} = e^{\pm i\ell\varphi}A_\ell^{\pm\ell}(\theta)$.

Then from $0 = L_\pm Y_\ell^{\pm\ell}$ we find:

$0 \stackrel{!}{=} \pm\hbar e^{\pm i\varphi}[\partial_\theta \pm i\cot\theta\,\partial_\varphi](e^{\pm i\ell\varphi}A_\ell^{\pm\ell}(\theta)) =$

$= \pm\hbar e^{\pm i\varphi(\ell+1)}[\partial_\theta - \ell\cot\theta]A_\ell^{\pm\ell}(\theta) \Rightarrow A_\ell^{\pm\ell}(\theta) = c_\ell \sin^\ell\theta$

Hence $Y_\ell^{\pm\ell}(\theta,\varphi) \sim e^{\pm i\ell\varphi}\sin^\ell\theta$.

2.6d) $(Y_\ell^{-\ell}(\pi-\theta, \varphi+\pi) = e^{-i\ell(\varphi+\pi)}\sin^\ell(\pi-\theta) = (-)^\ell Y_\ell^{-\ell}(\theta,\varphi)$

$L_+ \rightarrow \pm\hbar e^{+i(\varphi+\pi)}[\frac{\partial}{\partial(\pi-\theta)} + i\cot(\pi-\theta)\frac{\partial}{\partial(\varphi+\pi)}] =$

$= \pm\hbar(-)e^{i\varphi}[(-)\frac{\partial}{\partial\theta} + i(-)\cot\theta\frac{\partial}{\partial\varphi}] = L_+$

Hence $Y_\ell^m(\pi-\theta, \varphi+\pi) = (-)^\ell Y_\ell^m(\theta,\varphi)$.

27a) $$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \left| Y_\ell^m(\theta,\varphi)\right|^2 =$$

$$= \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta)^2 =$$

The integral independent of φ, hence the integral over φ gives a 2π factor.

$u = \cos\theta \quad du = |-\sin\theta|\, d\theta = \sin\theta\, d\theta \Rightarrow$

$$= \int_{-1}^{1} du \frac{2\ell+1}{2} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(u)^2 = 1 \quad ✓$$

27b)

$P_0^0(\cos\theta) = 1, \quad Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$

$$P_1^0(\cos\theta) = \frac{1}{2}\frac{d}{du}(u^2-1) = u = \cos\theta$$

$$P_1^1(\cos\theta) = \frac{(1-u^2)^{\frac{1}{2}}}{2} \frac{d^2}{du^2}(u^2-1)\Big| = (1-u^2)^{\frac{1}{2}}\Big|_{u=\cos\theta} = \sin\theta$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^1(\theta,\varphi) = -\sqrt{\frac{3}{4\pi}\frac{1}{2}}\sin\theta e^{i\varphi}$$
$$= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$$

$$P_2^0(\cos\theta) = \frac{d^2}{du^2}\frac{(u^2-1)^2}{2^2 2!}\Big| = \frac{d}{du}\left[\frac{2(u^2-1)2u}{8}\right]\Big| =$$

$$= \left[\frac{2(2u)^2 + 4(u^2-1)}{8}\right]\Big| = \frac{12\cos^2\theta - 4}{8} = \frac{1}{2}(3\cos^2\theta - 1)$$

$$P_2^1(\cos\theta) = \frac{(1-u^2)^{\frac{1}{2}}}{2^2 2!}\frac{d^3}{du^3}(u^2-1)^2\Big| = \frac{1}{8}(1-u^2)^{\frac{1}{2}}\frac{d^2}{du^2}\left[2(u^2-1)2u\right]\Big|$$

$$= \frac{1}{2}(1-u^2)^{\frac{1}{2}}\frac{d}{du}(3u^2-1)\Big| = 3(1-u^2)^{\frac{1}{2}}u = 3\cos\theta\sin\theta$$

$$P_2^2(\cos\theta) = \frac{1}{8}(1-u^2)\frac{d^4}{du^4}(u^2-1)^2\Big| = 3(1-u^2)\Big| = 3\sin^2\theta$$

$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{4\pi}}\frac{1}{2}(3\cos^2\theta - 1) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$Y_2^1(\theta,\varphi) = -\sqrt{\frac{5}{4\pi}\frac{1}{3!}}\, 3\cos\theta\sin\theta e^{i\varphi} = -\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{i\varphi}$$

$$Y_2^2(\theta,\varphi) = \sqrt{\frac{5}{4\pi}\frac{1}{(2+2)!}}\, 3\sin^2\theta e^{2i\varphi} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{i2\varphi}$$

27c) Note that $\psi(\theta,\varphi) = 2\sin\theta\cos\theta e^{i\varphi}$

$$= -2\sqrt{\frac{8\pi}{15}}\, Y_2^1(\theta,\varphi).$$