Wahrscheinlichkeitstheorie Übungsblatt 9

Aufgabe H 1.

Beweis. $x_{k,n} := \frac{x_k}{\sqrt{n}}$
$\to \mathbb{E}[1]\, X_{k,n} = 0$
$\to \sum_{k=1}^{n} \mathbb{E}[1]\, X_{k,n}^2 = \frac{1}{n} \sum_{k=1}^{n} 1 = 1$
$\to$ unabhängig
$\to$ L-B-Bedingung; $\forall \epsilon > 0$
$\sum_{k=1}^{n} \mathbb{E}[1]\, \frac{X_k^2}{n} \mathbb{1}_{\{|X_k| > \epsilon\sqrt{n}\}} \leq^{1} \sum_{k=1}^{n} \frac{1}{n\epsilon^\alpha \sqrt{n}^\alpha} \mathbb{E}[1]\, |X|^{2+\alpha}$
$\leq \sum_{k=1}^{n} \frac{1}{n\epsilon^\alpha \sqrt{n}^\alpha} \sup_k \mathbb{E}[1]\, |X_k|^{2+\alpha} \xrightarrow{n\to\infty} 0 \Rightarrow 0$ □

Aufgabe H 2.

Beweis.

- $$\phi_X(t) = \mathbb{E}[1]\, e^{itX} = \mathbb{E}[1]\, e^{it\frac{X+Y}{\sqrt{2}}} = \mathbb{E}[1]\, e^{i\frac{t}{\sqrt{2}}X} \mathbb{E}[1]\, e^{i\frac{t}{\sqrt{2}}Y} = \phi_X\left(\frac{t}{\sqrt{2}}\right) \phi_Y\left(\frac{t}{\sqrt{2}}\right)$$
  $$= \phi_X\left(\frac{t}{\sqrt{2}}\right)^2 = \ldots = \phi_X\left(\frac{X}{\sqrt{2}}\right)^{(2^n)}$$
- X_i iid $X_i \sim X$ $\frac{S_m}{\sqrt{m}} \xrightarrow{w} \mathcal{N}(0,\sigma)$
- $m = 2^n \Rightarrow \phi_{\frac{S_m}{\sqrt{n}}}(t) = \phi_{X_1}\left(\frac{t}{\sqrt{m}}\right) \cdots \phi_{X_m}\left(\frac{t}{\sqrt{m}}\right) = \phi_X\left(\frac{t}{\sqrt{m}}\right)^m = \phi_X\left(\frac{t}{\sqrt{2^n}}\right) \xrightarrow{2^n} \phi_{\mathcal{N}(0,\sigma)(t)}$
- $\phi_X(t) = \phi_{\frac{S_m}{\sqrt{m}}} \xrightarrow{m\to\infty} \phi_{\mathcal{N}(\prime,\infty)}(t)$

  $\Rightarrow X \sim \mathcal{N}(0,\sigma)$ □

Aufgabe H 3.

Beweis.

- $W(S_n) = \sum_{i=1}^{n} 2^{i2} = \sum_{i=1}^{n} 4^i - 1 = \frac{4^{n-1}-1}{4-1} - 10 = \frac{4^{n+1}-4}{3} \Rightarrow \frac{2^n}{\sqrt{W(S_n)}} = \frac{2^n\sqrt{3}}{\sqrt{4^{n+1}-4}} = \frac{2^n}{\sqrt{4^{n+1}-1}} \frac{\sqrt{3}}{2} \to \frac{\sqrt{3}}{2} \approx 0,86$
- $P(S_n > 2^n) \geq P(S_{n-2} + X_{n-1} + X_n > 2^n, X_{n+1} > 0, X_n > 0) \geq P(X_{n-1} > 0, X_n > 0) = \frac{1}{4}$
- $P\left(\frac{S_n}{\sqrt{W(S_n)}} \geq \frac{5}{6}\right) = P\left(S_n \geq \frac{5}{6}\sqrt{W(S_n)}\frac{2^n}{2^n}\right) = P\left(S_n \geq \frac{5}{6}\frac{\sqrt{W(S_N)}}{2^n}2^n\right) \geq P(S_n \geq 2) \geq \frac{1}{4}$
- $\mathcal{N}_{0,1}([\frac{5}{6},\infty]) = 1 - \mathcal{N}_{(0,1)}(] - \infty, \frac{5}{6}] < 1 - \mathcal{N}_{(0,1)(]-\infty,0,83]} < 1 - 0,79 = 0,21 < 0,25$

□

[1] da $\mathbb{E}[1]\, X_k^2 \epsilon^\alpha \sqrt{n}^\alpha \mathbb{1}_{\{|X_k|^\alpha > \epsilon^\alpha \sqrt{n}^\alpha\}} \leq \mathbb{E}[1]\, |X_k|^{2+\alpha}\, \mathbb{1}$